The Analytical Foundation of Mechanics of Discrete Systems in Lagrange's Théorie des Fonctions Analytiques, Compared with Lagrange's Earlier Treatement of This Topic

Marco Panza

The Analytical Foundation of Mechanics of Discrete Systems in Lagrange's *Théorie des Fonctions Analytiques*, Compared with Lagrange's Earlier Treatments of This Topic. Part 1[†],[*]

Marco PANZA[**]

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0. Introduction and Summary.

The epistemological attitude pervading Lagrange's *Théorie des fonctions analytiques*[1] is reductionism. Analysis is reduced to algebra; geometry is reduced to analysis; mechanics[2] is reduced to geometry. All mathematics is, then, reduced to algebra.

In another work,[3] I have tried to study the mathematical and philosophical purport of Lagrange's mathematical reductionism and its peculiarity, its assumptions and its consequences, along with the details of reduction of analysis to algebra. Here I would like to concentrate on Lagrange's reduction of mechanics to analysis.

This reduction is Lagrange's general research programme on the foundation of mechanics throughout his life. I will try to follow the evolution of this programme, from the early works to the *Théorie*, even if my chief aim is to study Lagrange's

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** Univ. de Genève, Enseignement d'Histoire et Philosophie des Sciences.

treatment of mechanics in the latter treatise.

If the goal of Lagrange's *Méchanique analitique* was to "réduire la théorie de cette science et l'art de résoudre les problêmes qui s'y rapportent, à des formules générales, dont le simple développement donne les équations nécessaires pour la solution de chaque problême"[4], the outline represented by the mechanical part of the *Théorie*[5] makes explicit—even if Lagrange does not emphasize it—that the principles of mechanics are mathematical propositions and not empirical generalisations.[6] As differential assumptions and concepts are here rejected and *calculus* is reformulated within a general theory of analytic functions, the whole mechanics is considered as a "branch" of this latter theory.

In the *Théorie,* Lagrange is concerned with mechanics only in order to show that his theory of analytical functions can be applied to it. Thus his aim is not to write a complete treatise of mechanics,[7] but only to expose a general method and to show how it works on some examples chosen among the most elementary and fundamental problems in mechanics. None of these problems is concerned with continuum mechanics and really the only part of mechanics Lagrange considers explicitly is the mechanics of discrete systems of mass-points[8] (I refer here to it simply by the expression "mechanics of discrete systems"). In spite of the importance, particularity and difficulty of XVIII-th century problems in continuum mechanics,[9] such a limitation is a very natural one from the point of view defended by Lagrange in the *Méchanique analitique*. According to him it should be possible, in fact, to solve these problems by intending them as problems concerning systems of infinite number of "particles" to which principles and methods of mechanics of discrete systems were applicable with some mathematical adaptations. The following quotation is a clear exposition of this view:

> Jusqu'ici nous avons considéré les corps comme des points; et nous avons vu comment on détermine les loix de l'équilibre de ces points, en quelque nombre qu'ils soient, et quelques forces qui agissent sur eux. Or un corps d'un volume et d'une figure quelconque, n'étant que l'assemblage d'une infinité de parties ou points matériels, il s'ensuit qu'on peut déterminer aussi les loix de l'équilibre des corps de figure quelconque, par l'application des principes précédens.
>
> En effet, la maniere ordinaire de résoudre les questions de Méchanique qui concernent les corps de masse finie, consiste à ne considérer d'abord qu'un certain nombre de points placés à des distances finies les uns des autres, et à chercher les loix de leur équilibre ou de leur mouvement; à étendre ensuite cette recherche à un nombre indéfini de points; enfin à supposer que le nombre des points devienne infini, et qu'en même tems leurs distances deviennent infiniments petites, et à faire aux formules trouvées pour un nombre fini de points, les réductions et les modifications que demande le passage du fini à l'infini.
>
> Ce procédé est, comme l'on voit, analogue aux méthodes géométriques et analitiques qui ont précédé le calcul infinitésimal; et si ce calcul a l'avantage

de faciliter et de simplifier d'une maniere surprenante, les solutions des questions qui ont rapport aux courbes, il ne le doit qu'à ce qu'il considere ces lignes en elles-mêmes, et comme courbes, sans avoir besoin de les regarder, premierement comme polygones, et ensuite comme courbes. Il y aura donc à peu-près la même avantage à traiter les problêmes de Méchanique dont il est question par des vois directes, et en considérant immédiatement les corps de masse finies comme des assemblages d'une infinité de points ou corpuscules, animés chacun par des forces données. Or rien n'est plus facile que de modifier et simplifier par cette considération, la méthode générale que nous venons de donner.[10]

A similar procedure is also proposed by Lagrange to determining the motion of a fluid mass:

> On pourrait—he writes—déduire immédiatement les loix du mouvement de[s] [...] fluides [incompressibles] de celles de leur équilibre [...]; car, par le Principe général [de la dynamique] [...] il ne faut qu'ajouter aux forces accélératrices actuelles, les nouvelles forces accélératrices $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$, dirigées suivant le coordonnés rectangles x, y, z.[11]
>
> [...] Mais nous croyons qu'il est plus conforme à l'objet de cet ouvrage d'appliquer directement aux fluides les équations générales données [...] pour le mouvement d'un systême quelconque de corps.
>
> [...] On peut considerer un fluide incompressible comme composé d'une infinité de particules qui se meuvent librement entre elles sans changer de volume.[12]

Not only has such a programme of reducing continuum mechanics to mechanics of discrete systems shown itself as largely illusory, but moreover the general idea that mechanics has to have the form of a system of equations deduced the one from the other according to a number of rules of symbolic transformation, can be understood as being opposed to the exigency of introducing or clarifying an opportune set of concepts, with the intention of giving a better explanation or a more profound description of *phenomena*, as well as presenting a solution for an ever-increasing number of problems in old and new domains.[13] However Lagrange's approach in foundation of mechanics is far from being an occasional or marginal programme in the context of XVIII-th century science and philosophy. It is, on the contary, one of the purest manifestations of the *esprit analytique* in its application to the mathematical sciences. My task is not to emphasize the obvious limits of Lagrange's programme relative to the evolution of mechanics as such, but to study the "pure mathematical" methods it employs in order to give to mechanics a "very analytical" form, and to show that their deep kernel rests substantially unchanged with the passage from the differential calculus to the theory of analytic functions. The general organisation of mechanics proposed by Lagrange in 1788 in the *Méchanique analitique* as soon as its methods and demonstrative patterns were in fact perfectly consistent with a programme of reduction of all mathematics to the polynomial algebra. To make clear such a limitation of my aim and to underline the intrinsic limits of Lagrange's approach, I shall speak of his mechanics as "mechanics of discrete systems". The questions concerning Lagrange's purpose of extension and the non-extensibility of this mechanics to the study of continuum bodies or fluids shall not be treated here.

In the first part of my work I shall sketch out the general organization and main methods and principles of Lagrange's three principal expositions of the foundation of mechanics of discrete systems before 1797: the two memoirs presented at the *Académie de Turin* (and published in 1760–61 volume) on the calculus of variations and its applications;[14] the two works on the libration of the moon of 1764 and 1780;[15] and, finally, the *Méchanique analitique* in 1788. This subject has been treated recently by C. Fraser[16] and, partly, by H. Pulte, W. Barroso Filho and C. Comte and A. Dahan-Dalmedico.[17] I would only like to underline some points with regard to my forthcoming analysis concerning the *Théorie des fonctions analytiques* and the mathematical tools used by Lagrange. My reason for reconsidering historical material which has already been analyzed is not my disagreement with previous studies, but the different perspective of my research. For all details I refer to the cited works.

In the second part I shall study Lagrange's foundation of mechanics of discrete systems in the *Théorie des fonctions analytiques*. I hope to be able to show a strict correspondence between Lagrange's attitude in this and his previous works.

Finally, in the third part very short, I shall put forward some comparative considerations and concluding remarks.

I. Outline of Lagrange's Expositions of a General Method of Mechanics of Discrete Systems before 1797.

1.

α. In the second volume of the *Mélanges de Philosophie et de Mathématiques de la Societé Royale de Turin*, dated 1760–61, the young Lagrange presents two memoirs which in fact constitute a single work. In the first Lagrange presents a new and much simplified version of the Eulerian calculus of variations; in the second he applies his new formalism to solve certain dynamical problems.

The origins of the first memoir and its relations with the epistolar correspondence between Euler and Lagrange have been recently studied by Fraser and Dahan.[18]

In Lagrange's version, the calculus of variations is a formal[19] procedure based on the introduction of a new differential "characteristic" δ different from d, generally acting upon variables and producing a new set of objects (as $\delta\varphi$, $\delta^2\varphi$, &c.) called "variations". It is based on the following two rules:

> R.1 even if δ and d express two different kind of (infinitesimal) increments, the algorithmic rules that govern d also govern δ; i.e.: even if "δZ exprimera une différence de Z qui ne sera pas la même que dZ", it "sera cependant formée par les mêmes régles";[20]

R.2 since δ and d express two independent operations,[21] their order can be inverted; i.e.: for all φ

(1) $$\delta(d\varphi) = d(\delta\varphi)$$

If Z is a function of the variables φ, ψ, ω, and of their differentials $d\varphi$, $d\psi$, $d\omega$; $d^2\varphi$, $d^2\psi$, $d^2\omega$; &c. and if we mark by $\delta\int_a^b Z$ the variation of $\int_a^b Z$, from R.1. it follows that the equation of *maximum* or *minimum* for the *formula*[22] $\int_a^b Z$ is:

(2) $$\delta\int_a^b Z = 0$$

that Lagrange considers as equivalent to:[23]

(3) $$\int_a^b \delta Z = 0$$

Here, the "function" Z has to be considered as a differential *formula*, in which the symbols φ, ψ, ω, $d\varphi$, $d\psi$, $d\omega$; &c. occur.[24] Thus $\int_a^b Z$ can represent a geometric or mechanical entity expressed in a differential system in three variables. The problem of the calculus of variations is to derive from (3) that the functional conditions for this quantity are an extreme. For R.1 and R.2, the derivation from (3) of a three-equation differential system (the solution of which expresses the derived functional relation) is simply formal.

β. Even if Lagrange in his memoir does not justify his formal procedure, it was probably suggested to him—as Fraser and Dahn argue—by the remark that in Eulerian geometric procedure the differentials come with two different meanings. Therefore the distinction between Euler's two different meanings of symbol d,[25] and the consideration of some isolated passages in Lagrange's subsequent works,[26] allow us to understand Lagrange's essential idea.

While looking for an extreme of a function we have to work on variable values linked by a specific functional relation, looking for an extreme of a functional, we have to work on independent variable values. Thus—if $\psi = \psi(\varphi)$ is a function expressed in a system of coordinates φ, ψ—while $d\varphi$ marks the elementary difference of the independent variable and $d\psi = \left(\frac{d\psi}{d\varphi}\right)d\varphi$, the functionally correlated difference of ψ, $\delta\varphi$ and $\delta\psi$ express *any elementary difference of the two variables* φ and ψ intended as independent of one from the other, i.e. without accounting for the functional link between them.[27] Therefore, the problem of looking for an extreme of a functional can be solved by expressing it as a differential *formula*, where the variables φ, ψ, &c.; $d\varphi$, $d\psi$, &c.; &c. occur, and by looking for the conditions that make the δ-differential of this *formula* equal to zero.[28]

For this purpose, we can exploit the independence between variations. First, we try to reduce the δ-differential *formula* to a polynomial form of type $A\delta\varphi + B\delta\psi + \&c. + C\delta d\varphi + D\delta d\psi + \&c. + \&c.$; second, we replace all variables by their values given by equations of condition[29] of the problem; and finally (by exploiting the algebraic method of indeterminate coefficients) we equate separately all the coefficients of the remaining independent variations to zero.

In order to apply this procedure to solve mechanical n-bodies problems it is necessary only to express the conditions of the problem by an equation like (2), to infer form it a δ-differential equation like

$$\text{(4)} \qquad \sum_{i=1}^{n} A_i\delta\varphi_i + B_i\delta\psi_i + C\delta\omega_i + \&c. = 0$$

and to replace the dependent variables by their expressions in independent variables.

The dependence between variables in (4) can be of two different types. First: if all variables φ_i, ψ_i, ω_i, &c. ($i = 1, 2, \ldots, n$) express the position of a body with respect to a common system of coordinates, the conditions of the problem may demand some constraints on positions of some bodies. Therefore, some variables have to be expressed in terms of other variables. This dependence is specifically relative to the particular mechanical problem that we are solving. Secondly: some variables in (4) can generally express some geometric or mechanical entity that we can represent by an analytical *formula* in terms of other variables. Thus, we cannot change the values of these last variables without changing the value to the first ones. This dependence is of a general order and comes from the formulation of the general mechanical equation (4).

If no variable in (4) depends on some other independent variables, (4) is the final equation of the problem and we can equate the coefficients of variations to zero.

Moreover, even if (4) is not the final equation of the problem (that is: some variables occurring in it depend on some other variable), we can try to express all its variables in terms of a set of completely independent variables (φ, ψ, ω, &c.) combining them with opportune values a_i, b_i, c_i, &c. ($i = 1, 2, \ldots, n$). In this way, we shall obtain an equation like

$$\text{(5)} \qquad A\delta\varphi + B\delta\psi + C\delta\omega + \&c. + \sum_{i=1}^{n} A_i da_i + B_i db_i + C_i dc_i + \&c. = 0$$

from which we can deduce again: $A = 0$, $B = 0$, $C = 0$, &c..

By this method, we can reduce all mechanics of discrete systems to:

i) a set of general principles;

ii) an analytical procedure to reduce these principles to an equation like (4);

iii) the following algebraic inference:

(6) if $\delta\varphi$, $\delta\psi$, $\delta\omega$, &c. are appropriate independent variations, then:

$$[A\delta\varphi + B\delta\psi + C\delta\omega + \delta c. = 0] \Rightarrow [A = 0, B = 0, C = 0, \&c.]$$

All Lagrange's works in mechanics of discrete systems before 1797 can be reduced to different versions and applications of this general method.

Thus, we may say that (6) is the essential formal rule of Lagrange's foundation of mechanics of discrete systems before 1797. I shall try to show how this general method works (independently of the use of the calculus of variations itself) in order to deduce all mechanics of discrete systems, from some general principle, both in the 1760–61 memoir and in the 1764 and 1780 memoirs about the libration of the moon and in *Méchanique analitique*. I shall limit myself to the mathematical tools of Lagrange's deduction.

γ. The goal of the second 1760–61 memoir is to show that "toutes les questions de Dynamiques"[30] can be solved easily by the application of the *calculus of variations* to the Eulerian *principle of least action*. By generalizing the Eulerian principle, Lagrange asserts:

(***Least action principle for discrete systems***)[31]

If n bodies (with mass) M_i $(i=1, 2, \cdots, n)$[32] mutually interact and are animated by central forces proportional to any function of the distance (of their point of applications from their origin) and we denote by s_i $(i=1, 2, \ldots, n)$ the spaces respectively covered by these bodies in a given time t and by v_i $(i=1, 2, \ldots, n)$ the respective speeds at the end of this time, then the *formula*

$$\sum_{i=1}^{n} M_i \int v_i ds_i \tag{7}$$

will always be a *maximum* or a *minimum*.[33]

Thus, to find the general equations of motion of a discrete system composed by the bodies (of mass) $M_1, M_2, \ldots, M_n$ we have to look for the functional condition derived from the equation

$$\delta \sum_{i=1}^{n} (M_i \int v_i ds_i) = 0 \tag{8}$$

that (M_i being constants and for $ds_i = v_i dt$ $(i=1, 2, \ldots, n)$) we can set in the form:

$$\int \left[\sum_{i=1}^{n} (M_i v_i \delta ds_i) + \sum_{i=1}^{n} (M_i v_i \delta v_i) dt \right] = 0 \tag{9}$$

The problem is to transform this equation into a related one of type

$$\int \sum_{i=1}^{n} M_i (A_i \delta\varphi_i + B_i \delta\psi_i + C_i \delta\omega_i + \&c.) = 0 \tag{10}$$

in which $\delta\varphi_i$, $\delta\psi_i$, $\delta\omega_i$, &c. are independent variations and the coefficients A_i, B_i, C_i &c. depend on the forces M_iP_i, M_iQ_i, &c. acting upon the bodies (of mass) M_i $(i=1, 2, \ldots, n)$.[34] For this purpose another "non analytical" principle is clearly necessary to introduce forces in their effective relations to speed. Starting from the equation of *conservation of vis viva*—i.e.

$$\sum_{i=1}^{n} (M_i v_i^2) = K - 2 \sum_{i=1}^{n} [M_i \int (P_i dp_i + Q_i dq_i + \&c.] - 2\Lambda \tag{11}$$

(where K is a constant depending on the primitive velocities of bodies p_i, q_i, &c. are the distances of the points of application of forces P_i, Q_i, &c. respectively from their origins and Λ is the sum of all the terms like $M_\mu M_\nu \int W_{\mu,\nu} dw_{\mu,\nu}$, given by the mutual attraction between the bodies of the system)—and exploiting R.1 and R.2, Lagrange arrives at the identity[35]:

$$\sum_{i=1}^{n} M_i v_i \delta v_i = -\sum_{i=1}^{n} M_i(P_i \delta p_i + Q_i \delta q_i + \&c.) - \delta\Lambda \tag{12}$$

To complete the transformation of equation (9), it is then necessary to chose a particular system of coordinates in which the differentials ds_i—as soon as the distances p_i, q_i, &c. and $w_{\mu,\nu}$ $(i, \mu, \nu = 1, 2, \ldots, n)$—can be expressed. If x, y and z are three orthogonal coordinates and x_i, y_i and z_i are their respective values determining the positions of the bodies in the considered discrete system, the differentials ds_i will be respectively equal to the radicals $\sqrt{dx_i^2 + dy_i^2 + dz_i^2}$ $(i = 1, 2, \ldots, n)$ and, according to (12), the equation (9) will take the form:[36]

$$\int \left\{ \sum_{i=1}^{n} M_i \left[d\left(v_i \frac{dx_i}{ds_i} \right) \delta x_i + d\left(v_i \frac{dy_i}{ds_i} \right) \delta y_i + d\left(v_i \frac{dz_i}{ds_i} \right) \delta z_i \right] + \left[\sum_{i=1}^{n} M_i(P_i \delta p_i + Q_i \delta q_i + \&c.) + \delta\Lambda \right] dt \right\} = 0 \tag{13}$$

The final step is, therefore, simply to express the variations of the distances p_i, q_i, &c. and $w_{\mu,\nu}$ in terms of the variations δx_i, δy_i, δz_i $(i, \mu, \nu = 1, 2, \ldots, n)$.

After these replacements we have a general final equation of type (10), therefore—to use Lagrange's words—

> si chaque corps est entiérement libre, en sorte que toutes les différences δx_1, δy_1, δz_1, δx_2, δy_2, &c. démeurent indéterminées, on fera chacun de leurs coéficiens $= 0$, et l'on aura trois fois autant d'équations qu'il y a de corps, lesquelles, prises ensemble sufiront pour déterminer toutes les vitesses, et les courbes cherchées: mais si un, ou plusieurs de ces corps sont forcés de se mouvoir sur des courbes, ou des surfaces données, et qu'ils agissent de plus, les uns sur les autres, soit en se poussant, soit en se tirant par des fils, ou des verges inflexibles, ou de quelque autre maniére que ce soit,[37] alors on cherchera les rapports qui devront nécessairement se trouver entre les différences δx_1, δy_1, δz_1, δx_2, δy_2, &c.. On réduira par là ces différences au plus petit nombre possible, et on fera ensuite chacun le leures coéficiens $= 0$, ce qui donnera toutes les équations nécessaires pour la solution du Problême.[38]

δ. Let us suppose the system to be completely free and the external forces to be reduced to $3n$ orthogonal forces $M_i X$, $M_i Y$, $M_i Z$ acting upon every body along directions parallel to the axes. If it is the case and we put $x_i = x + \xi_i$, $y_i = y + \eta_i$ and $z_i = z + \zeta_i$ (where ξ_i, η_i and ζ_i are new variables which express the position of points (x_i, y_i, z_i) $(i = 1, 2, \ldots, n)$ relatively to the point (x, y, z)), we can replace in (13) the variations δx_i, δy_i, δz_i $(i=1, 2, \ldots, n)$ with $\delta x+\delta\xi_i$, $\delta y+\delta\eta_i$, $\delta z+\delta\zeta_i$ and the sum $\mathrm{P}_i\delta p_i+\mathrm{Q}_i\delta q_i+$&c. with $\mathrm{X}(\delta x+\delta\xi_i)+\mathrm{Y}(\delta y+\delta\eta_i)+\mathrm{Z}(\delta z+\delta\zeta_i)$. In this way we shall have a new equation of type (5). Equating the coefficients of the independent variations δx, δy and δz in this equation separately to zero, we have the well-known equations of centre of mass of the system:

$$(14)\qquad \left\{\begin{aligned} &d\left(\frac{\sum_{i=1}^{n}\mathrm{M}_i dx_i}{dt}\right)+\sum_{i=1}^{n}\mathrm{M}_i\mathrm{X}dt=0\\ &d\left(\frac{\sum_{i=1}^{n}\mathrm{M}_i dy_i}{dt}\right)+\sum_{i=1}^{n}\mathrm{M}_i\mathrm{Y}dt=0\\ &d\left(\frac{\sum_{i=1}^{n}\mathrm{M}_i dz_i}{dt}\right)+\sum_{i=1}^{n}\mathrm{M}_i\mathrm{Z}dt=0 \end{aligned}\right.$$

(since the expressions of variations that occur in $\delta\Omega$ do not contain δx, δy, δz)[39], which Lagrange interprets as an expression of the properties of the centre of gravity of an arbitrary system of bodies: the point of coordinates

$$\left(\frac{\sum_{i=1}^{n}\mathrm{M}_i x_i}{\sum_{i=1}^{n}\mathrm{M}_i};\frac{\sum_{i=1}^{n}\mathrm{M}_i y_i}{\sum_{i=1}^{n}\mathrm{M}_i};\frac{\sum_{i=1}^{n}\mathrm{M}_i z_i}{\sum_{i=1}^{n}\mathrm{M}_i}\right)$$

—i.e. the centre of gravity of the system—"se mouvra comme ferait un corps sollicité simplement par les trois forces [M]X, [M]Y, [M]Z"[40]; this movement is, then, completely independent of the internal forces, that is, exactly, the Newtonian *principle of conservation of motion of the centre of gravity*.[41]

If we pass, now, from the orthogonal system of coordinates x, y, z to a cylindrical system ρ, θ, z, being ρ_i and θ_i respectively the values of the *radius* vectors and the corresponding angles determining the positions of the bodies in the discrete system considered, the differentials ds_i will be expressed by the radicals $\sqrt{\rho_i^2 d\theta_i^2+d\rho_i^2+dz_i^2}$ $(i=1, 2, \ldots, n)$. The substitution of this *formula* to ds_i in (9) and some other opportune transformations provide a new equation in cylindrical coordinates for the motion of a n-bodies discrete system. If such a system is supposed to be "entiérement libre" or it is "simplement assujetti à se mouvoir autour d'un point fixe" and "toutes les forces sollicitatrices des corps concourrent à ce point"[42] and we act on its equation as in the previous case, posing $\theta_i=\theta+\tilde{\theta}_i$ (where $\tilde{\theta}_i$ $(i=1, 2, \ldots, n)$ are new angular variables) and equating the coefficient of $\delta\theta$ to zero, we shall have, according to Lagrange:

$$(15) \qquad \sum_{i=1}^{n} \mathrm{M}_i d\left(v_i \frac{\rho_i^2 d\theta_i}{ds_i}\right) = 0$$

Thus, a simple integration leads to the equation of conservation of angular *momentum*:

$$(16) \qquad \sum_{i=1}^{n} \mathrm{M}_i v_i \frac{\rho_i^2 d\theta_i}{ds_i} \left[= \sum_{i=1}^{n} \mathrm{M}_i \rho_i \frac{\rho_i d\theta_i}{dt} \right] = \mathrm{W}$$

where W is a constant. Moreover, with a new integration we have

$$(17) \qquad \sum_{i=1}^{n} \mathrm{M}_i \int_0^t \rho_i^2 d\theta_i = \mathrm{W}t$$

Lagrange's historical remarks about (16) are the following:

> [...] nous remarquerons que l'équation (16) renferme le Principe que Mrs. Daniel Bernoulli et Euler ont appellé *la conservation du moment du mouvement circulatoire*, et qui consiste en ce que la somme des produits de chaque corps (M) par sa vitesse circulatoire $\left(\frac{v\rho d\theta}{ds}\right)$ et par sa distance au centre (ρ) est constante pendant le mouvement du sistème. [...]
>
> La même équation (16) renferme aussi le Principe de M. le Chevalier d'Arcy, que la somme des produits de chaque corps (M) par sa vitesse (v), et par la perpendiculaire $\left(\frac{\rho^2 d\theta}{ds}\right)$ menée du centre sur la direction du corps fait toujours une quantité constante.[43]

About equation (17) Lagrange notes that the integral $\int_0^t \rho_i^2 d\theta_i$ "exprime l'aire que l[...][es] projection[s] du corps M_i décri[...][vent] autour du centre des forces"[44]; so (17), combined with the corresponding equations deduced by taking x and y as linear coordinates in the cylindrical system, expresses the *law of areas*: the sum of the masses of every body multiplied by the area described by its *radius* vector around a fixed centre is directly proportional to time.[45]

ε. I resume. The second memoir of 1760–61 by Lagrange exposes a general method to reduce the general proposition expressing the principle of least action for a discrete system to a general equation of type (10), using the variational formalism and the assistance of the principle of conservation of *vis viva*. The algebraic method of indeterminate coefficients is, then, to be applied to this general equation to deduce analytically the equations of every body of the system (the correction of the general equation by the introduction of particular equations of condition is conceptually trivial).

To show the strength of his method, Lagrange derives from it the principle of conservation of motion of the centre of gravity in its Newtonian form, the principle of consevation of angular *momentum* and the law of areas for appropriate n-bodies discrete systems. These derivations are achieved by a *completely formal way* and do not depend on any physical or "metaphysical" intuition.

Even if Lagrange uses two different principles, the organization of the matter shows very well that he considers mechanics of discrete systems as an analytical deductive system based on the only principle of least action. Reference to the principle of conservation of *vis viva* is a necessary condition for transforming the analytical expression of the first principle to an appropriate variational equation.[46] This procedure is not, however, economic, as it requires two different principles. Therefore, even if Lagrange presents his foundation of mechanics as an analytical deduction from *one* principle, the realisation of this plan is not completely achieved.

2

ζ. Even if the organization of dynamics as a general theory proposed in 1760–61 was to be abandoned some years later by Lagrange himself, the general idea of reducing all mechanics to a formal analytical method that, applied to *one* general principle, is able to provide an equation of the type (10), from which to deduce the equations of motion of the system, by applying the general algebraic method of indeterminate coefficients, is Lagrange's lifelong research programme in the foundation of mechanics of discrete systems. Here, we can pick out a very general epistemological attitude which constitutes in fact, a philosophy of science.

In 1764, only three years after the memoir previously discussed, Lagrange's research programme changed in its internal organization.

Some conceptual reasons for this modification have been very well pointed out by Fraser.[47] I would simply add that the principle of virtual velocities, selected to provide the base of mechanics of discrete systems, has a very obvious interpretation as an analytical equation of the type (10). The introduction of another auxiliary principle is not necessary and the application of the general method is very natural. So, the choice of the new primitive principle can moreover be motivated by its consistence with Lagrange's mathematical method.

The opportunity to expose the new version of Lagrange's general method was the prize competition proposed for 1764 by the *Académie des Sciences de Paris* on the following subject:

> Si on peut expliquer par quelque raison physique pourquoi la Lune nous présente toujours à peu-près la même face; et comment on peut déterminer par les observations et par la théorie si l'axe de cette Planette est sujet à quelque mouvement propre, semblable à celui qu'on connaît dans l'axe de la terre, et qui produit la précession des équinoxes, et la mutation.[48]

Lagrange's winning memoir does not provide a complete and satisfactory theory of the libration of the moon, but it does contain an outline of a general method for dynamics of discrete systems. In the first paragraph Lagrange writes:

> Quoi qu'un très-grand Geomètre ait déja donné des méthodes et des formules générales, qui peuvent aisément s'appliquer à la recherche dont il s'agit ici, néanmoins il m'a paru plus commode de reprendre la question en

> entier, et de la résoudre par une méthode que je crois nouvelle à plusieurs égards, et qui est d'un usage simple et générale pour tous les Problêmes de Dynamique.[49]

The 1780 memoir in which Lagrange provides a more general and satisfactory theory of the libration of the moon to complete his previous studies was also the oppotunity to give a more elaborate and general version of his new general method.[50]

η. The origin of the new formulation of Lagrange's programme is clearly the *Traité de dynamique* of d'Alembert, issued in 1743, where dynamics is reduced to statics, according to the following principle (known as "*d'Alembert principle*"): if you communicate a motion to a system of bodies and if these bodies move according to another motion because of their mutual action, then you can consider the motion communicated as composed of the real motion of the bodies and of a virtual motion that can be considered destroyed.[51] Then it is clear that the destroyed motion balances the internal forces of that system.

The purpose is to find a simple reduction of dynamics of discrete systems to statics of these systems that does not require any decomposition of real motions.[52]

The starting point is a *generalization* of the classical *principle of virtual velocities* for discrete systems:

> (***Lagrange's generalised principle of virtual velocities for discrete systems***)
>
> Si un systeme quelconque de corps, réduits à des points, et tirés par des puissances quelconques, est en équilibre, et qu'on donne à ce systeme un petit mouvement quelconque en vertu duquel chaque corps parcourt un espace infiniment petit; la somme des puissances multipliées chacune par l'espace que le point où elle est appliquée parcourt suivant la direction de cette puissance, est toujours égale à zéro.[53]

Lagrange's general idea was to express the forces acting upon each body in the system, firstly as central foces[54] M_iP_i, M_iQ_i, M_iR_i, &c. directed along the lines p_i, q_i, r_i, &c., which mark the distances of the body from the origins of forces, and secondly as reduced to $3n$ orthogonal forces directed along parallel directions to three orthogonal axes x, y and z and then to consider the system of bodies subjected to both the former and the latter forces—considered as inversely directed—as a system in *equilibrium*, to which to apply the generalised statical principle of virtual velocities.

Lagrange does not explain what he means by the notions of force or vitrual velocity. He limits himself to translating the classic principle of virtual velocities (when forces are in *equilibrium*, the virtual velocities of bodies upon which they act—"evaluated along the directions of these forces"—are inversely as the forces themselves[55]) in previous "geometric" form. However, if we consider forces as causes of change in the state of bodies (in particular as cause of modification of motion),[56] we can consider virtual velocity as the natural uniform velocity of a body in absence of forces, i.e. as their initial velocity. This velocity is, then, naturally uniform and can be measured by a segment of a straight line and considered completely independent of the action of forces. Therefore, virtual velocity of a body can be geometrically expressed with an arbitrary vector starting from the given position of the body. Being independent of the forces (that is of the trajectory), it can be regarded as the result of the composition of three arbitrary and *completely independent orthogonal vectors* which we can represent by the variations δx_v, δy_v, δz_v $(1 \le v \le n)$ of orthogonal coordinates of the position of body. Thus, virtual velocities "evaluated along the direction" of a given force P_v can be represented by the variation $-\delta p_v$, where $p_v = p_v(x_v y_v z_v)$ is the distance between the body and origin of force. Therefore, Lagrange's general idea is there simply to equate the sums $-(\sum_{i=1}^{n} \mathrm{X}_i \delta x_i + \mathrm{Y}_i \delta y_i + \mathrm{Z}_i \delta z_i)$ and $(\sum_{i=1}^{n} \mathrm{P}_i \delta p_i + \mathrm{Q}_i \delta q_i + \&\text{c.})$, $\mathrm{M}_v\mathrm{X}_v$ $\mathrm{M}_v\mathrm{Y}_v$ and $\mathrm{M}_v\mathrm{Z}_v$ being, for every $v(v \in N,\ 1 \le v \le n)$, the projections on the axes of the vector expressing the total force acting upon the v-th body.

This interpretation of the principle is very natural and more general than the differential one, which uses only determinate d-differential differences representing only one of the infinite possible variations of a point. It is just this new generality that, by exploiting the respective independence of δx_i, δy_i and δz_i, allows to apply the algebraic method of indeterminate coefficients to the equation expressing the principle.

Really, representing the forces $\mathrm{M}_i\mathrm{X}_i$, $\mathrm{M}_i\mathrm{Y}_i$ and $\mathrm{M}_i\mathrm{Z}_i$ by their usual differential expressions, this equation assumes the general form:

$$\sum_{i=1}^{n} \mathrm{M}_i\left(\frac{d^2x_i}{dt^2}\delta x_i + \frac{d^2y_i}{dt^2}\delta y_i + \frac{d^2z_i}{dt^2}\delta z_i + \mathrm{P}_i\delta p_i + \mathrm{Q}_i\delta q_i + \&\text{c.}\right) = 0 \tag{18}$$

in which the distances p_i, q_i, &c. can be trivially expressed in terms of x_i, y_i, z_i, by expressing the origins of forces $\mathrm{M}_i\mathrm{P}_i$, $\mathrm{M}_i\mathrm{Q}_i$, &c. $(i = 1, 2, \ldots, n)$ by x, y, z coordinates. Thus, by exploiting R. 1, we can calculate δp_i, δq_i, &c. in terms of x_i, y_i, z_i, and, by simple replacements, the forces $\mathrm{M}_i\mathrm{P}_i$, $\mathrm{M}_i\mathrm{Q}_i$, &c., $(i = 1, 2, \ldots, n)$ which are considered functions of distances.

More generally, we can introduce some independent variables φ, ψ, ω, &c. in function of which we can express all the primitive variables x_i, y_i, z_i $(i = 1, 2, \ldots, n)$ for the equations of condition due to the mutual disposition of bodies (i.e.: we can look for some common and independent variables φ, ψ, ω, &c. in terms of which we can express respective positions of all bodies in the specific studied systems.) Thus (18) will take the form:

$$\Phi\delta\varphi + \Psi\delta\psi + \Omega\delta\omega + \&\text{c.} = 0 \tag{19}$$

from which we immediately deduce (for the independence of $\delta\varphi$, $\delta\psi$, $\delta\omega$, &c.):

(20)

$$\begin{cases} \Phi = 0 \\ \Psi = 0 \quad \Phi = \Phi(\varphi,\psi,\omega,\&\text{c.},\{k_i\});\ \Psi = \Psi(\varphi,\psi,\omega,\&\text{c.},\{k_i\});\ \Omega = \Omega(\varphi,\psi,\omega,\&\text{c.},\{k_i\}),\ \&\text{c.} \\ \Omega = 0 \\ \&\text{c.} \end{cases}$$

(where $\{k_i\}$ is a set of appropriate specific constants). The solution of this algebraic system gives the value of variables φ, ψ, ω, &c. and, therefore, the equations of motion of the system.

The formal consideration of x_i, y_i, z_i, as functions of φ, ψ, ω, &c. and the variational rules R.1 and R.2 give, *prior to any specific mechanical consideration*, some general transformation and operational rule able to simplify considerably the calculus necessary in order to apply the general method.

Among them, a special importance in the development of the method is assumed by the formal transformation of (18) into the equation:[57]

$$(21)\qquad \left[d\left(\frac{\delta \mathrm{T}}{\delta d\varphi}\right)-\left(\frac{\delta \mathrm{T}}{\delta\varphi}\right)+\left(\frac{\delta \mathrm{V}}{\delta\varphi}\right)\right]\delta\varphi+\left[d\left(\frac{\delta \mathrm{T}}{\delta d\psi}\right)-\left(\frac{\delta \mathrm{T}}{\delta\psi}\right)+\left(\frac{\delta \mathrm{V}}{\delta\psi}\right)\right]\delta\psi+\&\mathrm{c.}=0$$

in which we have to pose

$$(22)\qquad \begin{gathered} \mathrm{T}=\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\frac{\mathrm{M}_i}{2dt^2}(dx_i^2+dy_i^2+dz_i^2)\\ \mathrm{V}=\sum_{i=1}^{n}\beta_i=\sum_{i=1}^{n}\mathrm{M}_i\int \mathrm{P}_i dp_i+\mathrm{Q}_i dq_i+\&\mathrm{c.}\end{gathered}$$

and $\dfrac{\delta \mathrm{T}}{\delta\mu}=\sum_{i=1}^{n}\dfrac{\delta\alpha_i}{\delta\mu}$, $\dfrac{\delta \mathrm{T}}{\delta d\mu}=\sum_{i=1}^{n}\dfrac{\delta\alpha_i}{\delta d\mu}$, $\dfrac{\delta \mathrm{V}}{\delta\mu}=\sum_{i=1}^{n}\dfrac{\delta\beta_i}{\delta\mu}$ $(\mu=\varphi, \psi, \omega, \&\mathrm{c.})$ are respectively the sums of the coefficients of $\delta\mu$ and $\delta d\mu$ in $\delta\alpha_i$ $(i=1, 2, \ldots, n)$ and of $\delta\mu$ in $\delta\beta_i$ $(i=1, 2, \ldots, n)$.[58]

θ. The general deductive strength of the method is stressed by Lagrange both in the first and second memoir by a completely analytical deduction of the *principle of conservation of vis viva* for discrete systems, which can be considered a particular integral version of the generalised principle of virtual velocities for these systems. In fact, if the bodies of the system are considered to move, during the time dt through the infinitely small spaces ds_i (i.e.: if we fix the arbitrary value of the variation δs_i equal to differential ds_i) at speed v_i $(i=1, 2, \ldots, n)$, the general principle requires that:

$$(22)\qquad \sum_{i=1}^{n}\mathrm{M}_i\left(\frac{dv_i}{dt}ds_i+\mathrm{P}_i dp_i+\mathrm{Q}_i dq_i+\&\mathrm{c.}\right)=0$$

from which (11) (where Λ is equal to zero)[59] derives simply by putting $v_i=ds_i/dt$ and by integrating.

This is the deduction that Lagrange presents in his first memoir.[60]

The principle of conservation of *vis viva* is, however, only *one* of the possibly integral versions of the general principle of virtual velocities applied to dynamics of discrete systems. To express *vis viva* we have to consider virtual velocities as expressed by a differential and not, generally, by a variation. So, the general method of indeterminate coefficients cannot be applied to the general equation of conservation of *vis viva*, that is only a particular case of the equation of the principle of virtual velocity.

In his second memoir Lagrange underlines this fact and presents an alternative proof.[61] He makes the following remarks about it:

> Notre méthode donne ainsi une démonstration directe et générale de ce fameux Principe, mais on aurait tort de la confondre pour cela avec ce même Principe; car ce Principe ne donne de lui-même qu'une seule équation, et ne suffit seul que pour résoudre les problemes qui ne demandent qu'une seule équation; au lieu que notre méthode donne toujours toutes les équations nécessaires pour la solution du probleme.
>
> On aurait pu au reste déduire immédiatement le principe de la conservation des forces vives de l'équation générale [...][(18)] en y changeant la caractéristique δ en d (ce qui est évidemment permis, puisque les différences marquées par δ sont (indéterminées et arbitraires) et intégrant ensuite; mais nous avons cru qu'il n'étoit pas inutile de faire voir comment les différentes équations différentielles du mouvement du systeme fournissent toujours une équation intégrable, qui n'est autre chose que celle de la conservation des forces vives.[62]

This remark by Lagrange confirms that the heart of the new method (and its mathematical ingenuity and originality) is the use of arbitrary variations expressed as δ-differentials to allow general application of the method of indeterminate coefficients.

Lagrange's new proof in 1780 thus gives a more general interpretation of the principle of conservation of *vis viva* in a completely modified framework. It follows directly from the transformed form (21).

Since the variables φ, ψ, ω, &c. are all independent, we can build an algebraic system by separately equating the coefficients of $\delta\varphi$, $\delta\psi$, $\delta\omega$, &c. to zero. If we multiply the equations resulting respectively for $d\varphi$, $d\psi$, $d\omega$, &c. and we add them, we have (for $d\mu\cdot dv=d[\mu dv]-\mu d^2v$):

$$
\begin{aligned}
(23)\qquad & d\left[\left(\frac{\delta \mathrm{T}}{\delta d\varphi}\right)d\varphi+\left(\frac{\delta \mathrm{T}}{\delta d\psi}\right)d\psi+\&\mathrm{c.}\right]-\left[\left(\frac{\delta \mathrm{T}}{\delta d\varphi}\right)d^2\varphi+\left(\frac{\delta \mathrm{T}}{\delta d\psi}\right)d^2\psi+\&\mathrm{c.}\right]\\
& -\left[\left(\frac{\delta \mathrm{T}}{\delta \varphi}\right)d\varphi+\left(\frac{\delta \mathrm{T}}{\delta \psi}\right)d\psi+\&\mathrm{c.}\right]+\left[\left(\frac{\delta \mathrm{V}}{\delta \varphi}\right)\delta\varphi+\left(\frac{\delta \mathrm{V}}{\delta \psi}\right)\delta\psi+\&\mathrm{c.}\right]+\&\mathrm{c.}=0
\end{aligned}
$$

but, being $\mathrm{V}=\mathrm{V}(\varphi, \psi, \omega, \&\mathrm{c})$ and $\mathrm{T}=\mathrm{T}(\varphi, \psi, \omega, \&\mathrm{c}, d\varphi, d\psi, d\omega, \&\mathrm{c.})$ this equation is simply:

$$
(24)\qquad d\left[\left(\frac{\delta \mathrm{T}}{\delta d\varphi}\right)d\varphi+\left(\frac{\delta \mathrm{T}}{\delta d\psi}\right)d\psi+\&\mathrm{c.}\right]-d\mathrm{T}+d\mathrm{V}=0
$$

and so, by integrating, we have:

(25) $$\left(\frac{\delta \mathrm{T}}{\delta d\varphi}\right)d\varphi+\left(\frac{\delta \mathrm{T}}{\delta d\psi}\right)d\psi+\&\mathrm{c}.-\mathrm{T}+\mathrm{V}=\mathrm{C}$$

and thus:[63]

(26) $$2\mathrm{T}=\mathrm{K}-2\mathrm{V} \quad [\mathrm{K}=2\mathrm{C}]$$

or:

(27) $$\sum_{i=1}^{n}\mathrm{M}_i v_i^2=\mathrm{K}-2\sum_{i=1}^{n}\mathrm{M}_i\int \mathrm{P}_i dp_i+\mathrm{Q}_i dq_i+\&\mathrm{c}.$$

ι. If in a different mechanical framework, the second general version of Lagrange's dynamics of discrete systems is also an application of the very general method outlined above, in paragraph I. β.. It consists of a very formal procedure to deduce—from one general dynamical principle, interpreted in terms of variations—an opportune variational equation from which to deduce the equations of motion, by application of the algebraic method of indeterminate coefficients. Derivation of the principle of conservation of *vis viva* is now the proof of deductive strength and of the generality of the method.

But, even if it is essentially a variational procedure, dynamics of discrete systems can no longer be considered—as in the 1760–61 memoir—as a simple application of the calculus of variations to a general principle. We can even say that the calculus of variations is—in itself—absent from the general theory. If Lagrange bases his method on the introduction of new δ-differential variations, it does not depend on the solution of any problem of functional extreme.[64]

3

κ. The general idea of founding dynamics of discrete systems on the principle of virtual velocities and of generalising it by representing these velocities by δ-differentials in order to allow the application of the algebraic method of indeterminate coefficients, is also the basis of Lagrange's main scientific work: the *Méchanique analitique* published in 1788.[65]

In this work the general method laid down in the memoirs on the libration of the moon is generalised by the introduction of geometric methods and considerations, but remains fundamentally the same.

However, if in previous memoirs Lagrange was satisfied by showing as his method worked, now, in a general and very ambitious treatise, he had to discuss the conceptual and "metaphysical" foundations of his general principle. This need was even more pressing because of the essential epistemological goal of his work. As a matter of fact, Lagrange wanted to realize, explicitly and with all the necessary extension, his old research programme: to provide a reorganisation of mechanics of discrete systems according to an economic reductionist and analytical style,[66] founded on the introduction of δ-differentials and of the algebraic method of indeterminate coefficients. But this reductionism to analytic (algebraic) formalism—which is not only the constant attitude in Lagrange's entire scientific work, but also its profound reason and justification—is the only explicit philosophical thesis which Lagrange considered necessary to declare as a general attitude. On the contrary, he had a liking for historical remarks.

The matter is divided into two parts: *La Statique* and *La Dynamique*. In the first section of each part, Lagrange presents and discusses the general principles proposed to found mechanics in its history. However, he does not deal with the content of the famous discussion on the nature of force. His position has to be deduced from the choice and formulation of the general principle. In the second section of each part, he deduces, from the general selected principle, the "general *formulæ*" of statics and dynamics. In the third section of each part, he derives respectively the "general properties" of *equilibrium* and motion from these *formulæ* and once more finds the remaining general principles.[67] The following sections are devoted to more specific applications and to some mathematical developments.

λ. Statics is defined as "the science of *equilibrium* of forces". Even if forces do not really act in *equilibrium*, they produce "a tendency to motion",[68] and it has to be *measured* "by the effect it would produce if it were not arrested". If a force is considered a unity, the *expression* of every force is a ratio, "a mathematical quantity that can be represented by numbers or lines".[69]

In spite of his *ouverture*—according to which the laws of statics

> sont fondées sur des principes généraux qu'on peut réduire à trois: celui de *l'équilibre dans le levier*, celui de *la composition du mouvement*, et celui des *vitesses virtuelles* [...][70]

—in his text Lagrange exposes the mechanics of discrete systems as being entirely deduced from the only third principle which

> est non-seulement en lui-même très-simple et très-générale; il a de plus l'avantage précieux et unique de pouvoir se traduire en une formule générale qui renferme tous les problêmes qu'on peut proposer sur l'equilibre des corps.[71]

Perhaps in order to take out the ambiguity (which, from his point of view, is not considerable—the most important thing being the *deductive system*) in the second edition, Lagrange adds a new paragraph to the first section of the *Statique* which begins with the following words:

> Quant à la nature du principe des vitesses vîrtuelles, il faut convenir qu'il n'est pas assez évident par lui-même pour pouvoir être érigé en principe primitif; mais on peut le regarder comme l'expression générale des lois de l'équilibre, déduites des deux principes que nous venons d'exposer [the *principle of levers* and the *principle of composition of forces*]. Aussi dans les dèmonstrations qu'on a données de ce principe, on l'a toujours fait dépendre de ceux-ci, par des moyens plus ou moins directs. Mais il y a en Statique un autre principe général et indépendant du levier et de la composition des forces,

> quoique les mécaniciens l'y raportent communément, lequel paraît être le fondement naturel du principe des vîtesses virtuelles; on peut l'appeler le *principe des poulies.*[72]

The *principle of pulleys* is not, however, clearly expressed and the deduction of principle of virtual velocities is rather an intuitive justification.[73]

Such a principle is reconstructed by Mach in the following terms. Let the forces[74] M_iP_i act upon the mass-points of mass M_i ($i=1, 2, \ldots, n$) and let us suppose they had a common measure W/2, i.e. $M_iP_i=2\mu_i\frac{W}{2}$ (the constants μ_i being natural numbers). If two identical pulleys are respectively placed at each point at which the mass-points of mass M_i are placed and at the orgin of each force, the system of forces can be represented by an inextensible string to the end of which a weight W/2 is applied and which pass μ_1 times from point mass of mass M_1 and the origin of force M_1P_1, then μ_2 times from point-mass of mass M_2 and the origin of force $M_2P_2, \ldots,$ and μ_n times from point-mass of mass M_n and the origin of force M_nP_n.

> Or il est évidente—Lagrange writes—que, pour que le système tiré par ces différentes puissances demeure en équilibre, il faut que le poids ne puisse pas descendre par un déplacement quelconque infiniment petit des points du système [...].[75]

Let us consider such a little displacement as being virtually placed and let δp_i be the infinitely little spaces that the mass-points of mass M_i ($i=1, 2, \ldots, n$) would run along the directions of forces acting upon them because of this little movement. If such is the case, because of this displacement, the weight would run to the virtual space $\sum_{i=1}^{n} 2\mu_i\delta p_i$ corresponding to the difference of the length of the portion of string used to tie together the pulleys of the systems among them before and after the virtual displacement. As it is impossible that the weight goes on, if no other force act upon it, the condition of *equilibrium* of this system will be given by equating this virtual space to zero. By multiplying for the common measure W/2 we shall have:

$$\sum_{i=1}^{n} 2\mu_i\frac{W}{2}\delta p_i=\sum_{i=1}^{n} M_iP_i\delta p_i=0 \tag{28}$$

Lagrange argues that every system of forces can be considered a pulley system in such a way that the principle of virtual velocities is "proved, concerning commensurable powers". However, since

> on sait que toute proposition qu'on démontre pour des quantités commensurables, peut se démontrer également par la *réduction à l'absurde*, lorsque ces quantités sont incommensurables,[76]

the same principle will also be proved for incommensurable powers.

Lagrange's "proof" is quite unsatisfactory and recalls in its least step a metaphysical principle of numerical continuity.

μ. In whichever way the principle of virtual velocities is founded, it constitutes—if interpreted in terms of variations—a good general base for statics of discrete systems without the assistance of any other principle. This is, I think, a sufficient reason for Lagrange to accept it as *the* primitive principle of statics of discrete systems.[77]

It is now given by Lagrange in the following new formulation:

(***New formulation of principle of virtual velocities for equilibrium of discrete systems***)

Forces [acting upon a discrete system of points] are in *equilibrium* if they are in an inverse ratio to the virtral velocities of bodies upon which they act.

According to Lagrange, the virtual velocity of a body is the velocity which it is

disposé à recevoir, en cas que l'equilibre vienne à être rompu; c'est-à-dire la vitesse que ce corps prendroit réellement dans le premier instant de son mouvement.[78]

If the forces M_PP, M_QQ, M_RR, &c. are in *equilibrium* and p, q, r, &c. are any lines traced from the point where the body is placed and upon which the forces act and in the direction of these forces, the variations δp, δq, δr, &c., due to an infinitely small displacement of such body are directly proportional to virtual velocities and thus they can also measure them.[79] Two forces are in *equilibrium* only if their directions are opposite, so the principle asks for that:

(29) $$\frac{M_PP}{M_QP}=-\frac{dq}{dp}, \quad \text{that is:} \quad M_PP\delta p+M_QQ\delta q=0$$

If we consider *equilibrium* between three forces M_PP, M_QQ, M_RR, we can divide Q in two parts Q_1 and Q_2 such that Q_1 is in *equilibrium* with P. Thus for the *equilibrium* of the entire system we have:

$$M_PP\delta p+M_{Q_1}Q_1\delta q=0 \quad \text{and} \quad M_{Q_2}Q_2\delta q+M_RR\delta r=0$$

(30) that is:

$$M_PP\delta p+M_QQ\delta q+M_RR\delta r=0$$

Therefore the general condition of *equilibrium* is:

(31) $$M_PP\delta p+M_QQ\delta q+M_RR\delta r+\&c.=0$$

(as it was requested in the 1764 and 1780 formulation of the principle).

If we can the product $M_PP\delta p$—according to Galileo—the *momentum* of force M_PP, the principle says that the sum of the *momenta* of forces in *equilibrium* is always zero.

If the lines p, q, &c. are the distances from the origins of forces as in previous memoirs, their variations will be $-\delta p$, $-\delta q$, &c., but the general equation does not change.

If the origins of forces are movable, the variations δp, δq, &c. depend on both the variation of the point and the origin. By considering only one variation, we have a partial variation and the sum of the partial variations will give the total variation.

Following these considerations, Lagrange limits himself, in the first edition, to conclude that in the general *formulæ* of mechanics the variations have to be considered as total variations

> en regardant [...] comme variables toutes les quantités qui dépendent de la situation du systême, et comme constantes celles qui se rapportent aux points ou centres extérieurs [...].[80]

In the second edition, he introduces a general distinction between internal and external forces working explicitly throughout the book. It is however evident that the difference is only an expository one, this distinction being conceptually clear in the first edition as in Lagrange's earlier works.

If we consider a system of three orthogonal coordinates, and if (x, y, z) is the position of the body upon which the force M_PP acts and (a, b, c) is the position of its origin, we have:

$$p=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} \tag{32}$$

and, if the force is external, the origin being fixed:

$$\begin{aligned}\delta p=\delta_x p+\delta_y p+\delta_z p&=\frac{x-a}{p}\delta x+\frac{y-b}{p}\delta y+\frac{z-c}{p}\delta z\\&=\delta x\cos\alpha+\delta y\cos\beta+\delta z\cos\gamma\end{aligned} \tag{33}$$

while, if it is internal:

$$\begin{aligned}\delta p&=\delta_x p+\delta_y p+\delta_z p+\delta_a p+\delta_b p+\delta_c p\\&=\left(\frac{x-a}{p}\right)(\delta x-\delta a)+\left(\frac{y-b}{p}\right)(\delta y-\delta b)+\left(\frac{z-c}{p}\right)(\delta z-\delta c)\\&=(\delta x-\delta a)\cos\alpha+(\delta y-\delta b)\cos\beta+(\delta z-\delta c)\cos\gamma\end{aligned} \tag{34}$$

where α, β, γ are the angles formed by p (the direction of force M_PP) with the direction of axes (taken positively, starting from axes).[81]

Let δ_p be a short distance taken on the line marking the direction of force M_PP. $\delta_p=0$ being the condition which prevents the point from moving in any direction which is not perpendicular to this direction, this is, therefore, the differential equation of a surface perpendicular to the force M_PP. Thus the value of δp can be expressed starting from the differential equation of a perpendicular surface to the direction of MpP.[82]

By simple replacements we can come to the same results in terms of different types of coordinates, such as polar or cylindrical coordinates, in order to have the most appropriate form of the general equation relative to a particular problem and to search all the symmetries of the system.

When the equation of the *equilibrium* of an arbitrary system of forces acting upon a discrete system of points is given in terms of the selected system of coordinates, in order to have the equation of a particular system, we have to replace in it all the dependent coordinates with their expression in terms of independent coordinates we can derive from the equations of condition of the system. After having completed such a replacement, we shall have a new variational equation of the general form $\Phi\delta\varphi + \Psi\delta\psi + \Omega\delta\omega + \&c. = 0$, where the coefficients will be separately equated to zero. In his second edition, Lagrange exposes such a procedure in a very general manner.[83] If we express the coordinates of the points and of the origins of forces in terms of a number of independent variables φ, ψ, ω, &c. (according to the equations of condition), the segments p, q, r, &c. become functions of these variables and, for identity between d-derivatives and δ-derivatives, we have:

$$(35)\qquad \begin{aligned} &\delta p = \delta_\varphi p + \delta_\psi p + \&c. = \frac{\partial p}{\partial \varphi}\delta\varphi + \frac{\partial p}{\partial \psi}\delta\psi + \&c\,. \\ &\delta q = \delta_\varphi q + \delta_\psi q + \&c. = \frac{\partial q}{\partial \varphi}\delta\varphi + \frac{\partial q}{\partial \psi}\delta\psi + \&c\,. \\ &\&c. \end{aligned}$$

Thus, equating the coefficients of independent variations in (31) to zero we also have:

$$(36)\qquad \begin{cases} \Phi = \mathrm{M_P P}\left(\dfrac{\partial p}{\partial \varphi}\right) + \mathrm{M_Q Q}\left(\dfrac{\partial q}{\partial \varphi}\right) + \&c. = 0 \\ \Psi = \mathrm{M_P P}\left(\dfrac{\partial p}{\partial \psi}\right) + \mathrm{M_Q Q}\left(\dfrac{\partial q}{\partial \psi}\right) + \&c. = 0 \\ \&c. \end{cases}$$

that we can intend as the equations of all the "particular *equilibria*" composing the "general *equilibrium*" of the entire system.

Independently of the conditions of *equilibrium*, (35) implies:

$$(37)\qquad \mathrm{M_P P}\delta p + \mathrm{M_Q Q}\delta q + \&c. = \Phi\delta\varphi + \Psi\delta\psi + \&c.$$

which [if the variations $\delta\varphi$, $\delta\psi$, $\delta\omega$, &c. express the virtual velocities of the bodies upon which a number of forces expressed by Φ, Ψ, Ω, &c. act] expresses the equivalence between the system of forces $\mathrm{M_P P}$, $\mathrm{M_Q Q}$, $\mathrm{M_R R}$, &c. directed along the lines p, q, r, &c. and the system of forces expressed by Φ, Ψ, Ω, &c. directed along the lines φ, ψ, ω, &c.. So, the identical equations:

$$(38)\qquad \begin{aligned} &\mathrm{M_P P}\left(\frac{\partial p}{\partial \varphi}\right) + \mathrm{M_Q Q}\left(\frac{\partial q}{\partial \varphi}\right) + \&c. = \Phi \\ &\mathrm{M_P P}\left(\frac{\partial p}{\partial \psi}\right) + \mathrm{M_Q}\left(\frac{\partial q}{\partial \psi}\right) + \&c. = \Psi \\ &\&c. \end{aligned}$$

analytically express the *law of composition of forces*,[84] which Lagrange has thereby mathematically deduced.

ν. In the third part of the *Statique*, Lagrange shows how the principal properties of *equilibrium* of a system of bodies can be deduces starting from the general equation (31).[85]

In this way, he precisely states the mathematical connection of his principle of virtual velocities and his mechanical methods with the most usual mechanical principles, relative to the state of rest of any system of forces.

We shall consider some of these problems from the more general viewpoint of dynamics, Lagrange's procedures being substantially the same for both statics and dynamics. In fact, according to the general principle proposed in 1764 and 1780 memoirs, the general equation of dynamics of discrete systems differs from the general equation of statics for discrete systems only by the addition of new terms that also express *momenta* of forces.

ξ. *Equilibrium* is a relative condition. To determine the equation of *equilibrium* of a discrete system, we can consider forces and their effects only relatively. We can then be satisfied by noting forces with arbitrary functions of the distance between two points and representing their effects with virtual velocities measured by variations of these distances.

On the contrary, the general problem of dynamics of discrete systems is to study the actual motion of (punctual) bodies upon which same forces act. Thus, even if we can confine ourselves to considering the effects of forces, and not strictly their own nature, we still have to consider the modes of action of forces and we cannot confine ourselves to internal comparisons.

The only dynamic principles Lagrange considers explicitly are, however, those which govern the conduct of a system of bodies as such. While a principle of this type, as that of virtual velocities, can be enough to found a very "systemic" science such as statics, if we want to apply it to dynamic situations, it is necessary to express forces—or at least their effects—in analytical terms. Therefore, particular "non systemic" principles, which express the character of action of any force, are indispensable. Thus, "metaphysical" discussions cannot be completely avoided. Lagrange conceals the subject among historical remarks contained in the first section of part two and within "preliminary notions" in the first two paragraphs of section two.[86]

We can reconstruct Lagrange's presuppositions approximately in the following terms.

> La Dynamique est la science des forces accélératrices ou retardatrices, et des mouvements variés qu'elles doivent produire.[87]

These forces act to modify the natural uniform movement of each body. Their effect is continuous and so the speed of a body upon which a force acts changes continuously. The effect of force is modification of speed. To know a force is to know the speed it produces, starting form the given initial conditions (initial speed), during a given time (and in the absence of any resistance of the *medium*). If we consider the action of force during this time as uniform, the difference between final and initial speed—time being unitary—measures the force. But this is the real measure relative to every instant of time only if the force acts uniformly. If we want to measure the instantaneous effect of a non uniform force, we can consider a virtual uniform force acting during the given time and with the same value as the real one in the instant considered. Even if circular, this definition of instantaneous values of a force justifies the measure of the instantaneous effect of that by changing speed in an infinitely short time, during which any force can be considered as uniformly acting.[88] Because any speed can be measured by spaces run in a unitary time, we can consider the change of speed as a change of space. On the other hand, if speed is measured by a ratio between space and time, the effect of force will also be measured by a ratio between space and time. Force can thus be considered as a factor of modification of instantaneous velocity and expressed by a second derivative of space relative to time.

o. The general "systemic" principles Lagrange considers in the first section of his *Dynamique* are: the *principle of conservation of vis viva*; the *principle of conservation of motion of centre of gravity*; the *law of areas*; and the *least action principle*. The analytical deduction of these principles for discrete systems, in the third section, is a proof of the deductive power of the method (as in previous memoirs). This proof is now "complete": all alternative principles are deduced from that of *virtual velocities*.

The modes of application of this principle to dynamics are exactly the same as in the 1780 memoir. The general measure of forces by a second derivative of space relative to time is now justified by considerations contained in the last paragraph. However, to complete justification of (18), it is also necessary to assume:

$$\frac{d^2s}{dt^2}\delta s=\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z \tag{39}$$

Now, from (37) and (38) we can deduce:

$$\frac{d^2s}{dt^2}\delta s=\left(\frac{d^2s}{dt^2}\right)\left(\frac{\partial s}{\partial x}\right)\delta x+\left(\frac{d^2s}{dt^2}\right)\left(\frac{\partial s}{\partial y}\right)\delta y+\left(\frac{d^2s}{dt^2}\right)\left(\frac{\partial s}{\partial z}\right)\delta z \tag{40}$$

But $\frac{d^2x}{dt^2}$ is the projection of $\frac{d^2s}{dt^2}$ in a direction parallel to the x axis and thus, if α is the angle between direction of $\frac{d^2s}{dt^2}$ and axe x, we have $\frac{d^2x}{dt^2}=\left(\frac{d^2s}{dt^2}\right)\cos\alpha$. However for (33) $\cos\alpha=\frac{\delta_x s}{\delta x}$ and, for identity between d-derivatives and δ-derivatives, $\cos\alpha=\frac{\partial s}{\partial x}$, so $\frac{d^2x}{dt^2}=\left(\frac{d^2s}{dt^2}\right)\left(\frac{\partial s}{\partial x}\right)$, and (39) and (40) are therefore equivalent.[89]

Introducing masses and considering a system of n points upon which the forces M_iP_i, M_iQ_i, &c. act, we can immediately deduce (23), which also constitutes, the "general *formula* of dynamics" in the *Mécanique analytique*.

In the first edition, Lagrange emphasize how *formulæ* (32) and (33) allow to express variations δp, δq, &c. in terms of the variations δx, δy, δz. Therefore we can generally assume the formal identity:

$$\text{(41)} \qquad \mathrm{P}_i\delta p_i + \mathrm{Q}_i\delta q_i + \&\text{c.} = \mathrm{X}_i\delta x_i + \mathrm{Y}_i\delta y_i + Z_i\delta z_i \quad [i=1, 2, \ldots, n]$$

independently of the explicit determination of coefficients X, Y, Z according to (37) and (38).[90]

π. To prove the *principle of conservation of motion of centre of gravity*, the *law of areas* and the *conservation of vis viva*, Lagrange uses very similar procedures to those in the 1760–61 and 1764 memoirs.[91] Let the system be completely free. If we put: $x_i = x + \xi_i$; $y_i = y + \eta_i$; $z_i = z + \zeta_i$, it is clear that

> les quantités x, y, z n'entreront point dans les expression des distances mutuelles des corps [...]; par conséquent les équations de condition du systême seront entre les seul variables ξ_i, η_i, ζ_i et ne renfermeront point x, y, z.
>
> Donc si dans la formule générale du mouvement on substitue pour δx_i, δy_i, δz_i leurs valuers $\delta x + \delta\xi_i$, $\delta y + \delta\eta_i$, $\delta z + \mathrm{d}\zeta_i$, ces variations δx, δy, δz seront indépendantes de toutes les autres, et arbitraires en elles-mêmes.[92]

Thus, if in (18) we put $\mathrm{X}_i\delta x_i + \mathrm{Y}_i\delta y_i + \mathrm{Z}_i\delta z_i$ at place of $\mathrm{P}_i\delta p_i + \mathrm{Q}_i\delta q_i + \&\text{c.}$, according to (41), and replace δx_i, δy_i and δz_i with their previous values in terms of δx, δy and δz, equating the coefficients of these variations separately to zero, we have:

$$\text{(42)} \qquad \sum_{i=1}^{n} \mathrm{M}_i\left(\frac{d^2x_i}{dt^2} + \mathrm{X}_i\right) = 0\,; \quad \sum_{i=1}^{n} \mathrm{M}_i\left(\frac{d^2y_i}{dt^2} + \mathrm{Y}_i\right) = 0\,; \quad \sum_{i=1}^{n} \mathrm{M}_i\left(\frac{d^2z_i}{dt^2} + \mathrm{Z}_i\right) = 0$$

Now, if point (x, y, z) is the centre of gravity of the system, the sums $\sum_{i=1}^{n} \mathrm{M}_i\xi_i$, $\sum_{i=1}^{n} \mathrm{M}_i\eta_i$, $\sum_{i=1}^{n} \mathrm{M}_i\zeta_i$ are equal to zero[93] and, consequentially:

$$\text{(43)} \qquad \begin{aligned} \sum_{i=1}^{n} \mathrm{M}_i\frac{d^2x_i}{dt^2} &= \sum_{i=1}^{n} \mathrm{M}_i\left[\frac{d^2x}{dt^2} + \frac{d^2\xi_i}{dt^2}\right] = \frac{d^2x}{dt^2}\sum_{i=1}^{n}\mathrm{M}_i \\ \sum_{i=1}^{n} \mathrm{M}_i\frac{d^2y_i}{dt^2} &= \sum_{i=1}^{n} \mathrm{M}_i\left[\frac{d^2y}{dt^2} + \frac{d^2\eta_i}{dt^2}\right] = \frac{d^2y}{dt^2}\sum_{i=1}^{n}\mathrm{M}_i \\ \sum_{i=1}^{n} \mathrm{M}_i\frac{d^2z_i}{dt^2} &= \sum_{i=1}^{n} \mathrm{M}_i\left[\frac{d^2z}{dt^2} + \frac{d^2\xi_i}{dt^2}\right] = \frac{d^2z}{dt^2}\sum_{i=1}^{n}\mathrm{M}_i \end{aligned}$$

Thus (42) becomes:

$$\text{(44)} \qquad \begin{aligned} &\frac{d^2x}{dt^2}\sum_{i=1}^{n}\mathrm{M}_i + \sum_{i=1}^{n}\mathrm{X}_i\mathrm{M}_i = 0 \\ &\frac{d^2y}{dt^2}\sum_{i=1}^{n}\mathrm{M}_i + \sum_{i=1}^{n}\mathrm{Y}_i\mathrm{M}_i = 0 \\ &\frac{d^2z}{dt^2}\sum_{i=1}^{n}\mathrm{M}_i + \sum_{i=1}^{n}\mathrm{Z}_i\mathrm{M}_i = 0 \end{aligned}$$

which are the equations of motion of the centre of gravity in a completely free system. Hence, according to Lagrange,

> il est évident que le mouvement de ce centre ne dépendra point de l'action mutuelle que les corps peuvent exercer les uns sur les autres, mais seulement des forces accélératrices qui sollicitent chaque corps.[94]

For the deduction of the *law of areas*,[95] let us consider the system as completely free to turn around a fixed point, which we take as the origin of the axes. To express this we can compose the rotation around each axis. Let us consider first an infinitely small rotation of all the system around axis z. The most natural coordinates to express such a rotation are cylindrical coordinates ρ, θ, z—ρ being the *radius* vector and θ the corresponding angle. If the variations δx_i, δy_i, δz_i represent the infinitely corresponding displacements along the directions of three axes x, y and z of points of orthogonal coordinates (x_i, y_i, z_i), we shall have (because of the general identities $x_i=\rho_i \cos\theta_i$, $y_i=\rho_i \sin\theta_i$ and the obvious condition $\delta\rho=0$): $\delta x_i=-y_i\delta\theta_i$ and $\delta y_i=x_i\delta\theta_i$. But, since the system turns rigidly around axe z, the variations $\delta\theta_i$ of the angles θ_i $(i=1, 2, \ldots, n)$ are equal the one to each other and thus these identities take respectively the forms: $\delta x_i=-y_i\delta\theta$ and $\delta y_i=x_i\delta\theta$, where $\delta\theta$ is the "elementary" infinitely small rotation of the whole system around axis z. Composing these identities with the corresponding ones relative to infinitely small rotation of the system around axes y and x we shall have:

$$\delta x_i=z_i\delta\iota-y_i\delta\theta;\quad \delta y_i=x_i\delta\theta-z_i\delta\kappa;\quad \delta z_i=y_i\delta\kappa-x_i\delta\iota \tag{45}$$

(where $\delta\iota$ and $\delta\kappa$ are respectively the "elementary" infinitely small rotations of all the system around axes x and y) which express the orthogonal components of an infinitely small displacement of the point of orthogonal coordinates (x_i, y_i, z_i) according to a motion of rotation around the origin of the axes.

By making these replacements in (18), the *momenta* of the internal forces become null[96] so, by equating the coefficients of the independent variations $\delta\theta$, $\delta\iota$, $\delta\kappa$ to zero, after the replacements, we shall have:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \mathrm{M}_i\left[\frac{x_i d^2y_i-y_i d^2x_i}{dt^2}+\mathrm{Y}_i^e x_i-\mathrm{X}_i^e y_i\right]=0 \\ \displaystyle\sum_{i=1}^{n} \mathrm{M}_i\left[\frac{z_i d^2x_i-x_i d^2z_i}{dt^2}+\mathrm{X}_i^e z_i-\mathrm{Z}_i^e x_i\right]=0 \\ \displaystyle\sum_{i=1}^{n} \mathrm{M}_i\left[\frac{y_i d^2z_i-z_i d^2y_i}{dt^2}+\mathrm{Z}_i^e y_i-\mathrm{Y}_i^e z_i\right]=0 \end{cases} \tag{46}$$

(where X_i^e, Y_i^e and Z_i^e are as in the note (94)).

Thus, if all the external forces of the system are directed to the origin of the coordinates (i.e.: the motion of each body is due to composition of its initial velocity (virtual velocity) with a central force of which the origin is in the origin of the coordinates), the addenda $(\mathrm{Y}_i^e x_i-\mathrm{X}_i^e y_i)$, $(\mathrm{X}_i^e z_i-\mathrm{Z}_i^e x_i)$ and $(\mathrm{Z}_i^e y_i-\mathrm{Y}_i^e z_i)$ become zero[97] and so, by integrating relatively to t and multiplying by dt (being in general, $x_i = x_i(t)$; $y_i = y_i(t)$; $z_i = z_i(t)$):

$$\text{(47)} \qquad \begin{cases} \sum_{i=1}^{n} \mathrm{M}_i[x_i dy_i - y_i dx_i] = \mathrm{W}dt \\ \sum_{i=1}^{m} \mathrm{M}_i[z_i dx_i - x_i dz_i] = \mathrm{H}dt \\ \sum_{i=1}^{n} \mathrm{M}_i[y_i dz_i - z_i dy_i] = \mathrm{K}dt \end{cases}$$

where W, H and K are constants.

Now, if we consider the first equation and we replace x_i and y_i with their values in terms of cylindrical coordinates ρ, θ and z, we have:

$$\text{(48)} \qquad \sum_{i=1}^{n} \mathrm{M}_i[\rho_i \cos\theta_i d(\rho_i \sin\theta_i) - \rho_i \sin\theta_i \delta(\rho_i \cos\theta_i)] = \mathrm{A}dt$$

Because this equation expresses the projection of motion of the bodies on plane x, y, the system turning around axe z, the differential has to be taken relative to θ_i, and therefore (48) changes into (16), from which (17) derives integrating relatively to t.

The same reasoning being possible relative to other equations in (47), this differential system expresses the *law of areas*.

For the deduction of principle of conservation of *vis viva*, Lagrange repeats the same proof as in the first memoir on the libration of the moon. Changing δ for d in (18), if $\mathrm{P}_i dp_i + \mathrm{Q}_i dq_i + \&c.$ is integrable[98] and equal to $d\Pi_i$, by integrating we have:

$$\text{(49)} \qquad \sum_{i=1}^{n} \mathrm{M}_i\left(\frac{dx_i^2 + dy_i^2 + dz_i^2}{2dt^2} + \Pi_i\right) = \mathrm{K}$$

which expresses the principle.

To justify the replacement of d to δ, Lagrange writes:

> En général, de quelque maniere que les différens corps du systême soient disposés ou liés entre eux, pourvu que cette disposition soit indépendante du temps, c'est-à-dire, que les équations de condition entre les coordonnées ne renferment point la variable t; il est clair qu'on pourra toujours, dans la formule générale du mouvement, supposer les variations δx, δy, δz égales aux différentielles dx, dy, dz qui représentent les espaces effectifs parcourus par les corps dans l'instant dt, tandis que les variations dont nous parlons doivent représenter les espaces quelconques, que les corps pourroient parcourir dans le même instant, eu égard à leur disposition mutuelle.
>
> Cette supposition n'est que particuliere, et ne peut fournir par consequent, qu'une seule équation; mais étant indépendante de la forme du systême, elle a l'avantage de donner une équation générale pour le mouvement de quelque systême que ce soit.[99]

Lagrange's interpretation of the principle of *vis viva* for discrete systems is exactly the same as in 1780.

On the other hand, starting from (49), we can deduce the analytical expression of the *principle of least action* for discrete systems simply by applying variational formalism (i. e. R.1 and R.2).

If we note by v_i the speeds of the bodies of the system, δ-differentiating (49), if the equations of condition do not contain t, we have:

$$(50)\qquad \sum_{i=1}^{n} \mathrm{M}_i(v_i\delta v_i+\delta\Pi_i)=\sum_{i=1}^{n}\mathrm{M}_i[\mathrm{P}_i\delta p_i+\mathrm{Q}_i\delta q_i+\&\mathrm{c.}]+\sum_{i=1}^{n}\mathrm{M}_i[v_i\delta v_i]=0$$

Then (18) can be written in the form:

$$(51)\qquad \sum_{i=1}^{n}\mathrm{M}_i\left(\frac{d^2x_i}{dt^2}\delta x_i+\frac{d^2y_i}{dt^2}\delta y_i+\frac{d^2z_i}{dt^2}\delta z_i-v_i\delta v_i\right)=0$$

Now, by R.1 and R.2. we are able to deduce:

$$\begin{aligned}(52)\quad d^2x\delta x+d^2y\delta y+d^2z\delta z&=d[dx\delta x+dy\delta y+dz\delta z]-dxd\delta x-dyd\delta y-dzd\delta z\\&=d[dx\delta x+dy\delta y+dz\delta z]-dx\delta dx-dy\delta dy-dz\delta dz\\&=d[dx\delta x+dy\delta y+dz\delta z]-(1/2)\delta[dx^2+dy^2+dz^2]\\&=d[dx\delta x+dy\delta y+dz\delta z]-(1/2)\delta[ds^2]\\&=d[dx\delta x+dy\delta y+dz\delta z]-ds\cdot\delta ds\end{aligned}$$

Thus $\left(\text{for } dt^2=\dfrac{ds_i^2}{v_i^2}\right)$, (51) takes the form

$$(53)\qquad \sum_{i=1}^{n}\mathrm{M}_i\left[\frac{d[dx_i\delta x_i+dy_i\delta y_i+dz_i\delta z_i]}{dt^2}-\frac{v_i^2\delta ds_i}{ds_i}-v_i\delta v_i\right]=0$$

from which, multiplying by $dt=\dfrac{ds_i}{v_i}$ it follows that:

$$(54)\qquad \sum_{i=1}^{n}\mathrm{M}_i\left[\frac{d[dx_i\delta x_i+dy_i\delta y_i+dz_i\delta z_i]}{dt}-\delta[v_i ds_i]\right]=0$$

and d-integrating and inverting $\int$ and δ:

$$\sum_{i=1}^{n}\frac{\mathrm{M}_i[dx_i\delta x_i+dy_i\delta y_i+dz_i\delta z_i]}{dt}=\delta\left[\sum_{i=1}^{n}\mathrm{M}_i\int v_i\delta v_i\right]+\mathrm{K}$$

If the integral is taken as definite (as is implicit for Lagrange) and its limits are such that variations are null, both constant K and first member of (55) are null. Therefore, if the integral is taken between two fixed limits,[100] from (55) we deduce (8), and from here (for Lagrange's interpretation of δ), the *least action principle* for discrete systems, that Lagrange expresses in the following "geometric" form:

(***New formulation of principle of least action for discrete systems***)

[...] dans le mouvement d'un systême quelconque de corps animés par des forces mutuelles d'attraction, ou tendantes à des centres fixes, et proportionnelles à des fonctions quelconques des distances, les courbes décrites par les différents corps, et leurs vîtesses, sont nécessairement telles que la somme des produits de chaque masse par l'intégrale de la vîtesse multipliées par l'élément de la courbe est un *maximum* ou un *minimum*, pourvu que l'on regarde les premiers et les derniers points de chaque courbe comme donnés, en sorte que les variations des coordonnées répondantes à ces points soient nulles.[101]

To complete his discussion about relations between dynamic principles, Lagrange shows how we can, conversely, deduce the analytical expression of the principle of virtual velocities, starting from (8), resuming his 1760–61 method.

ρ. The first three sections of the two parts of *Méchanique analitique* are a complete and mature realisation of the general programme of 1764. The entire mechanics of discrete systems is reduced to the principle of virtual velocities, interpreted in terms of variations. This interpretation allows to deduce the equation of motion by a simple application of the algebraic method of indeterminate coefficients.

None of the central ideas working in these sections are new. The proofs of the main results are themselves more elegant reformulations of the previous ones. Calculus of variations is, in itself, considered only a mathematical tool to return from the principle of least action to that of virtual velocities.

On the contrary, completeness and generality are new. The general idea outlined in 1760–61 is now completely realized even touching a strictly modified form.

σ. As a concluding remark on Lagrange's variational foundation of mechanics of discrete systems on the principle of virtual velocities for discrete systems, we may observe that the representation of forces and speeds by spatial differences is quite enough to provide a "completely mathematical" deduction of the principle of virtual velocities itself, based on the simple argument that forces with the same directions add themselves like right segments.

Let us consider, for simplicity, only the case of an orthogonal three-coordinates system. The passage to other coordinate systems is trivial. Moreover, let t be a common parameter such that $x=x(t)$, $y=y(t)$, $z=z(t)$ and $dt=1$.

Let d^2x represent one force acting on a typical unitary mass-point placed at (x, y, z) and directed along a parallel direction to axis x. It might be considered a sum of a number of forces with the same direction acting upon the same mass-point. Thus, if we call these forces P_x, Q_x, &c. and we denote spatial differences produced by them by $d^2_{(\mathrm{P})}x$, $d^2_{(\mathrm{Q})}x$, &c., we have:

(56) $$d^2x = d^2_{(\mathrm{P})}x + d^2_{(\mathrm{Q})}x + \&c.$$

Speed is also represented by a spatial difference but, while force is a second difference, speed is a first one. Therefore, we can represent the virtual velocity of mass-point placed at (x, y, z) along the direction of axis x by a simple increment of x. But, while the spatial difference representing a force depends on the functional link between x and t, the virtual velocity of a mass-point is an intrinsic property of it and generally cannot be represented by dx. Let the virtual velocity along the direction of axis x be represented by any arbitrary increment of x, which, according to Lagrange, we mark by δx.

We take δx as a positive increment; if spatial difference produced by virtual velocity decreases the value of x, we mark it by $-\delta x$. It is a matter of definition to say that the *momentum* of a force is product of spatial difference representing force multiplied by spatial difference representing virtual velocity of the mass-point upon which the force acts, "evaluated along direction of force" (and oriented like it). Thus, for the *momentum* of force represented by d^2x we have:

(57) $$d^2x\delta x = d^2_{(\mathrm{P})}x\delta x + d^2_{(\mathrm{Q})}x\delta x + \&\mathrm{c}.$$

Let d^2p, d^2q, &c. represent some forces acting upon the mass-point placed at (x, y, z) and directed along the arbitrary directions of lines p, q, &c. in three dimensional x, y, z space. If these forces are such that the projections of spatial differences d^2p, d^2q, &c. on a line parallel to axis x passing through the point (x, y, z) are just $d^2_{(\mathrm{P})}x$, $d^2_{(\mathrm{Q})}x$, &c., we may consider these forces to be producing spatial differences which, "evaluated along a parallel direction to x axis", correspond to spatial differences produced by P_x, Q_x, &c. Let us note these forces by P, Q, &c..

Now, because d^2p, d^2q, &c. can be considered segments of straight lines p, q, &c. and $d^2_{(\mathrm{P})}x$, $d^2_{(\mathrm{Q})}x$, &c. as segments of x axis, if α_p, α_q, &c. are the angles that p, q, &c. form with the x axis, it is geometrically obvious that:

(58) $$d^2_{(\mathrm{P})}x = d^2p\cos\alpha_p\,; \quad d^2_{(\mathrm{Q})}x = d^2q\cos\alpha_q\,; \quad \&\mathrm{c}.$$

On the other hand, if p, q, &c. are considered respectively as the distances between the point (x, y, z) and the origins of forces P, Q, &c., we have

(59) $$\cos\alpha_p = \frac{\partial p}{\partial x}\,; \quad \cos\alpha_Q = \frac{\partial q}{\partial x}\,; \quad \&\mathrm{c}.$$

and therefore, replacing in (57):

(60) $$d^2x\delta x = d^2p\left(\frac{\partial p}{\partial x}\right)\delta x + d^2q\left(\frac{\partial q}{\partial x}\right)\delta x + \&\mathrm{c}.$$

Then, reasoning in the same manner relative to y and z coordinates, we can deduce:

(61) $$d^2x\delta x + d^2y\delta y + d^2z\delta z = d^2p\left[\left(\frac{\partial p}{\partial x}\right)\delta x + \left(\frac{\partial p}{\partial y}\right)\delta y + \left(\frac{\partial p}{\partial z}\right)\delta z\right] + d^2q\left[\left(\frac{\partial q}{\partial x}\right)\delta x + \left(\frac{\partial q}{\partial y}\right)\delta y + \left(\frac{\partial q}{\partial z}\right)\delta z\right] + \&\mathrm{c}.$$

Now, if the virtual velocity of the mass-point placed at (x, y, z) along the direction of axis x is represented by the difference between $x+\delta x$ *and* x, the virtual velocity of the masspoint placed at (x, y, z) "evaluated along direction p" will be represented by the difference between p and the distance between $(x+\delta x, y+\delta y, z+\delta y)$ and the origin (a_P, b_P, c_P) of the force P. Thus:

$$\delta p = \sqrt{(x-a_P)^2+(y-b_P)^2+(z-c_P)^2} - \sqrt{(x+\delta x-a_P)^2+(y+\delta y-b_P)^2+(z+\delta z-c_P)^2} \tag{62}$$

and, by developing:

$$\begin{aligned}\delta p = & -\frac{x-a_P}{p}\delta x - \frac{y-b_P}{p}\delta y - \frac{z-c_P}{p}\delta z + \mathrm{A}dx^2 + \mathrm{B}\delta y^2 + \mathrm{C}\delta z^2 \\ & + \mathrm{D}\delta x\delta y + \mathrm{E}\delta x\delta z + \mathrm{F}\delta y\delta z + \&\mathrm{c.} \\ = & -\left(\frac{\partial p}{\partial x}\right)\delta x - \left(\frac{\partial p}{\partial y}\right)\delta y - \left(\frac{\partial p}{\partial z}\right)\delta z + \mathrm{A}\delta x^2 + \mathrm{B}\delta y^2 + \&\mathrm{c.}\end{aligned} \tag{63}$$

If we consider δx, δy, δz, as infinitely small increases, we omit the infinitely small quantities of higher order and we repeat the same reasoning for q, &c., replacing in (61) we have:

$$d^2x\delta x + d^2y\delta y + d^2z\delta z + d^2p\delta p + d^2q\delta q + \&\mathrm{c.} = 0 \tag{64}$$

Introducing indexes, masses and time, we have exactly (18).

Thus the principle of virtual velocities is a purely *mathematical* consequence of differential interpretation of speed and force.

If we are able to provide a functional non-differential interpretation of speed and force, and we replace the virtual velocities δx, δy, δz, &c. with indeterminate arbitrary increaments of the variables and the virtual velocities δp, δq, &c. with the first increments of p, q, &c. relative to these increments of x, y, z,[102] we may even give a completely mathematical non differential foundation of mechanics of discrete systems.

For this, a new and non differential interpretation of the calculus of variations is not even necessary. Lagrange's method is, in itself, completely independent of it, and, further, of any specifically variational formalism. Only the *mathematical* relations between virtual velocities which this formalism expresses are, it fact, necessary to a correct application of the general method.

Therefore the complete reduction of mechanics of discrete systems to algebra depends *only* on the reduction of geometry and differential calculus to algebra. The interpretation of mechanics of discrete systems given by Lagrange in the *Méchanique analitique* perfectly accords with this radical reductionist programme.

Notes

1. Cf. Lagrange (1797) [1st edition] and (1813) [2nd edition].

2. Cf. *ibid.*, p. 223 and 311:

   [...] on peut regarder la mécanique comme une géométrie à quatre dimensions, et l'analyse mécanique comme une extension de l'analyse géométrique.
3. Cf. Panza (1992), part. III, chap, VI.
4. Cf. Lagrange (1788), p. V; cf. also Lagrange (1811–15), t. I, p. V.
5. Cf. Lagrange (1797), pp. 223–77 and (1813), pp. 311–81.
6. The question whether mechanical laws are mathematical (necessary) or empirical (contingent) is one of the main questions in the philosophy of mechanics in the XVIII-th century. I have discussed some aspects of this question in my (*forth.*), especially in paragraph 1; cf. also Dhombres and Radelet-De Grave (1991).
7. Cf. Lagrange (1797), p. 276 and (1813), p. 381.
8. Following Lagrange's terminology I shall use the term "body" to refer to what we call now "mass-point". Thus a body will be considered as placed at one point and forces acting upon it will be considered as acting upon such a point.
9. For an outline of a reconstruction of the history of mechanics in XVIII-th century in which the continuum mechanics takes its role of an essential (and really not separate) part of mechanics cf. Truesdell (1960b).
10. Cf. Lagrange (1788), pp. 50–1 and (1811–15), t, I, pp. 79–80. In the pages that follow this passage Lagrange develops a general method to found the conditions of *equilibrium* of a cotinuum body he applies successively to the research of the conditions of *equilibrium* of a string, a solid body and of a fluid mass [cf. *ibid.*, pp. 89–157 and pp. 136–220].
11. Cf. the next paragraph I. η..
12. Cf. Lagrange (1788), pp. 437–38 and (1811–15), t. II, pp. 286–87.
13. Truesdell has insisted with a particular force on these points in a number of papers; cf., for example, Truesdell (1954), p. CXXV, (1960a), pp. 409–12, (1960b), pp. 33–5 and (1967), p. 250 [cf. also Dahan-Dalmedico (1990), p. 100].
14. Cf. Lagrange (1760–61a) and (1760–61b).
15. Cf. Lagrange (1764) and (1780).
16. Cf. Fraser (1983) [cf. also Fraser (1985a)].
17. Cf. Pulte (1989), Barroso Filho and Comte (1988) and Dahan-Dalmedico (1990).
18. Cf. Fraser (1985a), pp. 155–72 and Dahan-Dalmedico (1990), pp. 81–88; cf. also Goldstine (1980), pp. 110–29.
19. The terms "formal", "formalism", &c. are among which have most varying meaning in the history and philosophy of mathematics. I shall use them, as regards eighteenth century mathematics, simply to refer to chains of symbolic deductions based on a set of explicit rules, concerning general relations between symbols. The recurrence of similar procedures in Lagrange's mathematical practice does not correspond to a philosophical attitude reducing all mathematics (and mechanics) to a mere symbolic game or to the acceptance of a protohilbertian programme.
20. Cf. Lagrange (1760–61a), pp. 174.
21. This "justification" of rule 2 will be made explicit by Lagrange only in Lagrange (1780), p. 218. In the London Royal Society Library copy of the first edition of the *Théorie* there are eight "sheets of notes and calculation in a contemporary hand" [cf. *Book Catalogue of the Library of the Royal Society*, by A. J. Clark, Univ. Publ. of America, vol. III (1982), p. 342]. Sheet number 8 contains the following particularly interesting remark (of which the author and the date is unknown):

    Z = fonction de x, y, z, ... dx, dy, dz, ... d^2x, d^2y, d^2z, ... rendre $\int Z$ maximum ou minimum entre des limites désignés. Soit t la variable implicite dont x, y, z, ... sont censées des fonctions [...]. Il faut donc que ces fonctions soient telles que si on les remplaçait dans $\int Z$ par d'autres, comme $x+\delta x$, $y+\delta y$, $z+\delta z$, composées de celles là et d'un accroissement δx, δy, δz, ... le résultat de l'integration (entre deux limites assignées) fut toujours plus g.e ou toujours plus petit que dans le 1^er cas. Or des que la fonction x ou y ... change de forme, il est clair que les coiffici. diff. en changent aussi et que ce qu'on représentoit par dx doit l'être à présent par $d(x+\delta x) = dx + d\delta x$.

Donc la quantité $d\delta x$ est l'accroissement de la fontion dx et comme l'accr. d'une foncion u d'une variable nous l'avons représenté par δu, on doit pareillement représentér par δdx l'accroissement de la fonction quelconque dx, d'où il suit $\delta dx = d\delta x$.

In other words, if we take the functional symbol F as an operator (or the operator d as a functional symbol) and we pose, by definition, $\delta[\mathrm{F}(\varphi)] = \mathrm{F}(\varphi + \delta\varphi) - \mathrm{F}(\varphi)$, we have, in the particular case when $\mathrm{F}(\varphi) = d\varphi$, $\delta d\varphi = d(\varphi + \delta\varphi) - d\varphi = d\delta\varphi$. R.2 is therefore a consequence of R. 1. This justification of the second rule is perfectly consistent with Lagrange's *concept* of variation as I shall try to explain.

22. Lagrange does not explain the limits of integration and writes $\int$ for $\int_a^b$. However, from the context it is clear that he refers to definite integrals the limits of which are not generally constants, but are determined by the specifically geometric or analytic conditions of the problem.
23. Making the simplest example ($\mathrm{Z} = \mathrm{Z}(\varphi)$) and putting $\dfrac{d\mathrm{X}(\varphi)}{d\varphi} = \mathrm{Z}$, we have: $\delta\int\mathrm{Z} = \dfrac{1}{d\varphi}\delta X(\varphi)$ and (according to R.1):
$$\int\delta\mathrm{Z} = \int\delta\left(\frac{d\mathrm{X}(\varphi)}{d\varphi}\right) = \int\left(\frac{d^2\mathrm{X}(\varphi)}{d\varphi^2}\right)\delta\varphi = \frac{1}{d\varphi}\frac{d\mathrm{X}(\varphi)}{d\varphi}\delta\varphi = \frac{1}{d\varphi}\delta\mathrm{X}(\varphi) = \delta\int\mathrm{Z}$$
24. The function Z really must be a particular function of its variables, according to some condition of homogeneity.
25. Cf. Fraser (1985a), p. 158 and Dahan-Dalmedico (1990), p. 83. The reference is of course to Euler (1744).
26. Cf., for example, Langrange (1788), p. 51.
27. In the general case in which ψ is a not linear function of φ, we shall therefore have: $d^2\varphi = 0$, $\delta^2\varphi = 0$, $d^2\psi = d(\psi'(\varphi)d\varphi) \neq 0$ and $\delta^2\psi = 0$.
28. Clearly, if φ and ψ are, in themselves, dependent on a principal variable t (which is eliminated in the functional equation between φ and ψ), the variations $\delta\varphi = \varphi(t+\delta t) - \varphi(t)$, $\delta\psi = \psi(t+\delta t) - \psi(t)$ (where δt is an infinitesimal arbitrary increaments) are respectively nothing but the increments of φ and ψ corresponding (according to functional links $\varphi = \varphi(t)$ and $\psi = \psi(t)$) to arbitary increment δt, *i.e.*:
$$\delta\varphi = \frac{d\varphi}{dt}\delta t + \frac{d^2\varphi}{dt^2}\frac{\delta t^2}{2!} + \&\mathrm{c.} = \frac{d\varphi}{dt}\delta t$$
$$\delta\psi = \frac{d\psi}{dt}\delta t + \frac{d^2\psi}{dt^2}\frac{\delta t^2}{2!} + \&\mathrm{c.} = \frac{d\psi}{dt}\delta t$$
which justify R.1. Analogously the variation $\delta\omega = \omega(\varphi + \delta\varphi, \psi + \delta\psi) - \omega(\varphi, \psi)$ of a function $\omega = \omega(\varphi, \varphi)$ is nothing but the increment of ω corresponding to arbitrary and independent increments $\delta\varphi$ and $\delta\psi$. Thus the difference between differentials and variations lies in the reference variable in differentiation. Cf. also the note 102.
29. Here the equations of condition have to be intended as equations expressing particular constraints to the motion of bodies due to the internal configuration of the system.
30. Cf Lagrange (1760–61b), p. 196.
31. Lagrange presents the Eulerian principle in a very generalized form, but without a previous discussion of the conditions of its application. This is a natural consequence of Maupertuis' and Euler's approaches that, even if different, are both based on the consideration of the principle of least action as a general law of nature [cf. on this point Panza (*forth.*)]. It is rather the passage from the principle to the final equations of motions of a system that may require some restrictions (see, as an example, the note (35)). Here, I have preferred to follow Lagrange in his faith on generality, rather then to point out all the restrictions of his formalism we know today.
32. We clearly have to intend Lagrange's bodies as mass-points [cf. the note (8)].

33. Cf. Lagrange (1760–61b), p. 196. The notation is not, of course, the original one by Lagrange. In this work, however, I shall change the original notations every time this change involves a simplification without being in my opinion unfaithful to the spirit of the original mathematical procedure. Thus, the changes in notations are part of my historical interpretation.
34. Really, Lagrange denotes forces only by P_i, Q_i, &c. (he calls "accelerative forces") introducing the masses only in the general equations of the system as common factors for which "forces" have to be multiplies. To conform my language to the usual one, in all of my paper I shall call "forces" the products of factors as P_i, Q_i, &c.—expressing the "absolute intensity" of these forces—for the masses of bodies upon which such forces act.
35. δ-differentiating the two members of (11) we have, in fact

$$\sum_{i=1}^{n} M_i v_i \delta v_i = -\sum_{i=1}^{n}\left[M_i \delta \int (P_i dp_i + Q_i dq_i + \&c.)\right] - \delta \Lambda$$

For R.1 and R.2. we have, also:

$$\delta \int P dp + Q dq + \&c. = \int \delta(P dp) + \delta(Q dq) + \&c. = \int \delta P dp + P\delta(dp) + \delta Q dq + Q\delta(dq) + \&c.$$

$$= \int \delta P dp + P d(\delta p) + \delta Q dq + Q d(\delta q) + \&c.$$

$$= \int \delta P dp + \int P d(\delta p) + \int \delta Q dq + \int Q d(\delta q) + \&c.$$

and, integrating by part the second, the fourth, &c. integral:

$$\delta \int P dp + Q dq + \&c. = P\delta p + Q\delta q + \&c. + \int \delta P dp - dP\delta p + \delta Q dq - dQ\delta q + \&c.$$

Now, if P is a function of p, Q of q, &c. from R. 1 it follows:

$$dP = P' dp \Rightarrow \delta P = P' \delta p$$

$$dQ = Q' dq \Rightarrow \delta Q = Q' \delta q$$

&c.

and so:

$$\delta P dp = dP \delta p (= P' \delta p dp = P' dp \delta p)$$

$$\delta Q dq = dQ \delta q (= Q' \delta q dq = Q' dq \delta q)$$

&c.

or:

$$\delta \int P dp + Q dq + \&c. = P\delta p + Q\delta q + \&c.$$

In this deduction the conditions $P = P(p)$, $Q = Q(q)$, &c. are essential. In a successive remark Lagrange observes, however, that this condition is not strictly necessary for his result. In fact, if P, Q, &c. are considered as functions of all variables p, q, &c., we have:

$$dP = \frac{\partial P}{\partial p} dp + \frac{\partial P}{\partial q} dq + \&c.\,; \quad \delta P = \frac{\partial P}{\partial p} \delta p + \frac{\partial P}{\partial q} \delta q + \&c.$$

$$dQ = \frac{\partial Q}{\partial p} dp + \frac{\partial Q}{\partial q} dq + \&c.\,; \quad \delta Q = \frac{\partial Q}{\partial p} \delta p + \frac{\partial Q}{\partial q} \delta q + \&c.$$

&c. &c.

and so:

$$\delta P dp - dP\delta p + \delta Q dq - dQ\delta q + \&c. = \left(\frac{\partial P}{\partial q} - \frac{\partial Q}{\partial p}\right)(dp\delta q - dq\delta p) + \left(\frac{\partial P}{\partial r} - \frac{\partial R}{\partial p}\right)(dp\delta r - dr\delta p) + \&c.$$

that is equal to zero if

$$\frac{\partial P}{\partial q} = \frac{\partial Q}{\partial p}; \quad \frac{\partial P}{\partial r} = \frac{\partial R}{\partial p}; \quad \&c.$$

what is ture if

$$P dp + Q dq + \&c. = dF \quad [F = F(p, q, r, \&c.)]$$

So, the only condition Lagrange thinks necessary on P, Q, R, &c. to derive (12) is that $Pdp + Qdq + Rdr + \&c.$ is an exact differential, that is an integrable multivariational function. The laws of motions deduced from Lagrange's method are therefore proved for any kind of forces, assuming that $Pdp + Qdq + \&c.$ is an exact differential.

36. Substituting ds_i with its expression in the orthogonal coordinates x, y, z, we shall have, according to R.1,

$$\int \sum_{i=1}^{n} (M_i v_i \delta ds_i) = \sum_{i=1}^{n} M_i \left[\int v_i \frac{dx_i}{ds_i} d\delta x_i + \int v_i \frac{dy_i}{ds_i} d\delta y_i + \int v_i \frac{dz_i}{ds_i} d\delta z_i \right]$$

and, integrating by parts:

$$\int v \frac{d\alpha}{ds} d\delta\alpha = v \frac{d\alpha}{ds} \delta\alpha - \int d\left(v \frac{d\alpha}{ds}\right)\delta\alpha \quad [\alpha = x_i, y_i, z_i; \ i = 1, 2, \ldots, n]$$

in which, if the position of the body at the beginning and at the end of time t is given, the first term of the second member is zero, "car cette supposition fera évanoüir les premiers et les derniers δx, δy, δz" [cf. *ibid.*, pp. 200].

37. It is evident that among these mutual actions we do not have to consider internal attractive forces, which are cinsidered as central forces.
38. Cf. Lagrange (1760–61b), pp. 208–09.
39. Cf. the note (94).
40. Cf. *ibid.*, p. 211. Cf. the note (34).
41. Cf. (Newton 1687), p. 17.
42. Cf. Lagrange (1760–61b), p. 212.
43. Cf. *ibid.*, p. 213–14; cf. also Lagrange (1788), pp. 185–88. Lagrange refers to D. Bernoulli (1745), Euler (1746) and d'Arcy (1749) and (1752). On Lagrange's historical references in the context of the history of the law of *moment of momentum* cf. Truesdell (1964), in particular pp. 594–97 and 600–02.
44. Cf. Lagrange (1760–61b), p. 213.
45. Lagrange refers to d'Arcy (1747).
46. Lagrange's use of the *vis viva* principle is very formal. In his first memoir, Lagrange–given the equation of condition for a *maximum* or a *minimum* of the integral *formula* $\int Z$, that is $\delta\int Z = \int\delta Z = 0$ – exploits the general form of total differential to write directly, according to R.1. (being $Z = Z\,(x, y, z, dx, dy, dz, d^2x, d^2y, d^2z, \&c.)$):

$$\delta Z = H_0\delta x + H_1\delta dx + H_2\delta d^2x + \&c. + K_0\delta y + K_1\delta dy + K_2\delta d^2y + \&c. + J_0\delta z + J_1\delta dz + J_2\delta d^2z + \&c.$$

where H_0, H_1, H_2, &c.; K_0, K_1, K_2, &c.; J_0, J_1, J_2, &c. are indeterminate coefficients. This formal equality is the base of Lagrange's general method. Applying it to deduce the analytical consequences of the principle of least action, Lagrange has to write the quantity $v\delta v$ in the form of a total differential, v being considered as a function of an opportune set of variables. The *vis viva* principle allows, in Lagrange's procedure, exactly this formal transformation.

47. Cf. Fraser (1985a), pp. 233–35.
48. Cf. Lagrange (1764), p. 3.
49. Cf. *ibid.*, p. 1–2.
50. Cf. Lagrange (1780), section I, pp. 213–24.
51. Cf. d'Alembert (1743), pp. 50–1. On *d'Alembert's principle* cf. Fraser (1985b) and Truesdell (1960a), pp. 186–88. I discuss d'Alembert's approach in Panza (*forth.*), paragraph 3.
52. In his introduction of (1780), Lagrange writes [cf. *ibid.*, p. 209]:

    La premiere [section] est destinée à l'exposition d'une méthode générale et analytique pour résoudre tous les problemes de la Dynamique. Cette méthode, que j'ai employée le premier dans ma Pièce sur la Libration de la Lune, a l'avantage singulier de ne demander aucune construction ni aucun raisonement géométrique ou mécanique, mais seulement des opérations analytiques assujetties à une marche simple et uniforme. Elle n'est autre chose que le Principe de Dynamique de M. d'Alembert, réduit en formule au moyen du Principe de l'équilibre appelé communément *loi des vitesses virtuelles*.

53. Cf. Lagrange (1780), p. 213. Cf. also Lagrange (1764), p. 5.
54. As in his 1760–61 memoir, Lagrange denotes forces simply by P_i, Q_i, R_i, &c.. Cf. the note (34).
55. Lagrange's references [cf. Lagrange (1764), p. 6] are to Varignon (1725), vol. 2, pp. 174 and foll. (where a letter of Johann Bernoulli of 26/1/1711 is quoted) and to "derived" principles of Maupertuis (1740) and Euler (1751) [pp. 195–97] (which I study in Panza (*forth.*), paragraphs 2 and 8). In the *Méchanique analitique* Lagrange adds to these references that of Cortivaron (1748–49) [cf. Lagrange (1788), p. 11]. In the following I shall use Lagrangian expression "virtual velocity evaluated along direction of force [M]X" meaning what J. Bernoulli, in his letter to Varignon, called "virtual velocity of force [M]X" [cf. Varignon (1725), p. 175]:

    Concevez [...] plusieurs forces différentes qui agissent suivant différentes tendances ou directions [...]; concevez aussi que l'on imprime à tout le systême de ces forces un petit mouvement, soit parallel à soi-mème suivant une direction quelconque, soit autour d'un point fixe quelconque: il vous sera aisé de comprendre que par ce mouvement chacun de ces forces avancera ou reculera dans sa direction, à moins que quelqu'un ou plusieurs des forces n'ayent leurs tendances perpendiculaires à la direction du petit mouvement; au quel cas cette force, ou ces forces n'avanceroient ni ne reculeroient de rien: car ces avancements ou recoulements, qui sont ce que j'appelle *vîtesses virtuelles*, ne sont autre chose que ce dont chaque ligne de tendance augmente ou diminue par le petit mouvement, & ces augmentations ou diminutions se trouvent, si l'on tire une perpendiculaire à l'extremité de la ligne de tendances de quelque force, laquelle perpendiculaire retranchera de la même ligne de tendance, mise dans la situation voisine par le petit mouvement, une petite partie que sera la mesure de la *vîtesse virtuelle* de cette force.

56. Cf. Euler's definition 10 in his *Mechanica* [cf. Euler (1736), vol. 1, p. 39]:

    Potentia est vis corpus vel ex quiete in motum perducens, vel motum eius alterans.

57. Cf. Lagrange (1780), pp. 218–20.
58. The proof of this standard and *a priori* transformation clearly shows both the formal attitude of Lagrange and the algorithmic relations between d and δ involved by R.1 and R.2. It is interesting to reconstruct it on the basis of Lagrange's indications. If we put:

$$\alpha = \frac{M}{2dt^2}(dx^2 + dy^2 + dz^2)\,; \quad \beta = M\int(Pdp + Qdq + \&c.)$$

$x = F(\varphi, \psi, \omega, \&c.)$; $y = G(\varphi, \psi, \omega, \&c.)$; $z = H(\varphi, \psi, \omega, \&c.)$; $Pdp + Qdp + \&c. = \chi(\varphi, \psi, \omega, \&c.)$; we immediately have (according to R.1):

$$\delta\alpha=\frac{\mathrm{M}}{dt^2}(d\mathrm{F}\delta d\mathrm{F}+d\mathrm{G}\delta d\mathrm{G}+d\mathrm{H}\delta d\mathrm{H})$$

$$=\frac{\mathrm{M}}{dt^2}\left[\begin{array}{l} d\mathrm{F}\left(\delta\left(\frac{\partial \mathrm{F}}{\partial\varphi}\right)d\varphi+\left(\frac{\partial \mathrm{F}}{\partial\varphi}\right)\delta d\varphi+\delta\left(\frac{\partial \mathrm{F}}{\partial\psi}\right)d\psi+\left(\frac{\partial \mathrm{F}}{\partial\psi}\right)\delta d\psi+\&\mathrm{c.}\right)\\ d\mathrm{G}\left(\delta\left(\frac{\partial \mathrm{G}}{\partial\varphi}\right)d\varphi+\left(\frac{\partial \mathrm{G}}{\partial\varphi}\right)\delta d\varphi+\delta\left(\frac{\partial \mathrm{G}}{\partial\psi}\right)d\psi+\left(\frac{\partial \mathrm{G}}{\partial\psi}\right)\delta d\psi+\&\mathrm{c.}\right)\\ d\mathrm{H}\left(\delta\left(\frac{\partial \mathrm{H}}{\partial\varphi}\right)d\varphi+\left(\frac{\partial \mathrm{H}}{\partial\varphi}\right)\delta d\varphi+\delta\left(\frac{\partial \mathrm{H}}{\partial\psi}\right)d\psi+\left(\frac{\partial \mathrm{H}}{\partial\psi}\right)\delta d\psi+\&\mathrm{c.}\right)\end{array}\right]$$

$$\delta\beta=\mathrm{M}\int\delta[\chi(\varphi,\ \psi,\ \omega,\ \&\mathrm{c.})]\quad \text{(by inverting } \textstyle\int \text{ and } \delta)$$

$$=\mathrm{M}\left[\delta\varphi\int\left(\frac{\partial\chi}{\partial\varphi}\right)+\delta\psi\int\left(\frac{\partial\chi}{\partial\psi}\right)+\&\mathrm{c.}\right]$$

But (still because of the equality between δ-derivative and d-derivative) we also have:

$$\delta\left(\frac{\partial \mathrm{F}}{\partial\varphi}\right)=\left(\frac{\partial^2 \mathrm{F}}{\partial\varphi^2}\right)\delta\varphi+\left(\frac{\partial^2 \mathrm{F}}{\partial\psi\partial\varphi}\right)\delta\psi+\left(\frac{\partial^2 \mathrm{F}}{\partial\omega\partial\varphi}\right)\delta\omega+\&\mathrm{c.}$$

$$\delta\left(\frac{\partial \mathrm{F}}{\partial\psi}\right)=\left(\frac{\partial^2 \mathrm{F}}{\partial\varphi\partial\psi}\right)\delta\varphi+\left(\frac{\partial^2 \mathrm{F}}{\partial\psi^2}\right)\delta\psi+\left(\frac{\partial^2 \mathrm{F}}{\partial\omega\partial\psi}\right)\delta\omega+\&\mathrm{c.}$$

&c.

and the same with G and H. Thus, if we note, according to Lagrange, by $(\delta\alpha/\delta\varphi)$, $(\delta\alpha/\delta d\varphi)$, $(\delta\beta/\delta\varphi)$; $(\delta\alpha/\delta\psi)$, $(\delta\alpha/\delta d\psi)$, $(\delta\beta/\delta\psi)$; &c. respectively: the coefficients of $\delta\varphi$ and $\delta d\varphi$ in $\delta\alpha$ and of $\delta\varphi$ in $\delta\beta$; the coefficients of $\delta\psi$ and $\delta d\psi$ in $\delta\alpha$ and of $\delta\psi$, in $\delta\beta$; &c. we have:

$$\left(\frac{\delta\alpha}{\delta\varphi}\right)=\frac{\mathrm{M}}{dt^2}\left[\begin{array}{r} d\mathrm{F}\left[\left(\frac{\partial^2\mathrm{F}}{\partial\varphi^2}\right)d\varphi+\left(\frac{\partial^2\mathrm{F}}{\partial\varphi\partial\psi}\right)d\psi+\&\mathrm{c.}\right]+\\ +d\mathrm{G}\left[\left(\frac{\partial^2\mathrm{G}}{\partial\varphi^2}\right)d\varphi+\left(\frac{\partial^2\mathrm{G}}{\partial\varphi\partial\psi}\right)d\psi+\&\mathrm{c.}\right]+\\ +d\mathrm{H}\left[\left(\frac{\partial^2\mathrm{H}}{\partial\varphi^2}\right)d\varphi+\left(\frac{\partial^2\mathrm{H}}{\partial\varphi\partial\psi}\right)d\psi+\&\mathrm{c.}\right]+\end{array}\right]$$

$$=\frac{\mathrm{M}}{dt^2}\left[d\mathrm{F}\cdot d\left(\frac{\partial\mathrm{F}}{\partial\varphi}\right)+d\mathrm{G}\cdot d\left(\frac{\partial\mathrm{G}}{\partial\varphi}\right)+d\mathrm{H}\cdot d\left(\frac{\partial\mathrm{H}}{\partial\varphi}\right)\right]$$

$$\left(\frac{\delta\alpha}{\delta\psi}\right)=\frac{\mathrm{M}}{dt^2}\left[d\mathrm{F}\cdot d\left(\frac{\partial\mathrm{F}}{\partial\psi}\right)+d\mathrm{G}\cdot d\left(\frac{\partial\mathrm{G}}{\partial\psi}\right)+d\mathrm{H}\cdot d\left(\frac{\partial\mathrm{H}}{\partial\psi}\right)\right]$$

&c.

$$\left(\frac{\delta\alpha}{\delta d\varphi}\right)=\frac{\mathrm{M}}{dt^2}\left[d\mathrm{F}\cdot\left(\frac{\partial\mathrm{F}}{\partial\varphi}\right)+d\mathrm{G}\cdot\left(\frac{\partial\mathrm{G}}{\partial\varphi}\right)+d\mathrm{H}\cdot\left(\frac{\partial\mathrm{H}}{\partial\varphi}\right)\right]$$

$$\left(\frac{\delta\alpha}{\delta d\psi}\right)=\frac{\mathrm{M}}{dt^2}\left[d\mathrm{F}\cdot\left(\frac{\partial\mathrm{F}}{\partial\psi}\right)+d\mathrm{G}\cdot\left(\frac{\partial\mathrm{G}}{\partial\psi}\right)+d\mathrm{H}\cdot\left(\frac{\partial\mathrm{H}}{\partial\psi}\right)\right]$$

&c

$$\left(\frac{\delta\beta}{\delta\varphi}\right)=\mathrm{M}\int\frac{\partial\chi}{\partial\varphi}$$

$$\left(\frac{\delta\beta}{\delta\psi}\right)=\mathrm{M}\int\frac{\partial\chi}{\partial\psi}$$

&c.

But:

$$\left(\frac{\partial\beta}{\partial\varphi}\right)\delta\varphi+\left(\frac{\partial\beta}{\partial\psi}\right)\delta\psi+\&\text{c.}=\delta\beta=\delta\left[\int\mathrm{P}dp+\int\mathrm{Q}dq+\&\text{c.}\right]$$

and, if we put:

$$\int\mathrm{P}dp=\mathrm{I}_1(p)\,;\quad\int\mathrm{Q}dp=\mathrm{I}_2(q)\,;\quad\&\text{c.}$$

then (still because of equalities between d-derivatives and δ-derivatives):

$$\delta\beta=\delta[\mathrm{I}_1(p)+\mathrm{I}_2(q)+\&\text{c.}]=\delta\mathrm{I}_1(p)+\delta\mathrm{I}_2(q)+\&\text{c.}=\mathrm{I}_1'(p)+\mathrm{I}_2'(q)+\&\text{c.}=\mathrm{P}\delta p+\mathrm{Q}\delta q+\&\text{c.}$$

that is:

$$\left(\frac{\partial\beta}{\partial\varphi}\right)\delta\varphi+\left(\frac{\partial\beta}{\partial\psi}\right)\delta\psi+\&\text{c.}=\mathrm{P}\delta p+\mathrm{Q}\delta q+\&\text{c.}$$

So, by differentiating $(\delta\alpha/\delta d\varphi)$, $(\delta\alpha/\delta d\psi)$, &c., and adding, we can simply verify that:

$$\left[d\left(\frac{\delta\alpha}{\delta d\varphi}\right)-\left(\frac{\delta\alpha}{\delta\varphi}\right)+\left(\frac{\delta\beta}{\delta\varphi}\right)\right]\delta\varphi+\left[d\left(\frac{\delta\alpha}{\delta d\psi}\right)-\left(\frac{\delta\alpha}{\delta\psi}\right)+\left(\frac{\delta\beta}{\delta\psi}\right)\right]\delta\psi+\&\text{c.}$$
$$=\mathrm{M}\left(\frac{d^2x}{dt^2}\delta x+\frac{d^2y}{dt^2}\delta y+\frac{d^2z}{dt^2}\delta z+\mathrm{P}\delta p+\mathrm{Q}\delta q+\&\text{c.}\right)$$

which (by the introduction of indices, substitutions (22) and consideration of (18)) corresponds to (21).

59. In the new context $\mathrm{M}_i\mathrm{P}_i$, $\mathrm{M}_i\mathrm{Q}_i$, &c. ($i=1, 2, \ldots, n$) express, in fact, *all* the forces acting upon the bodies (independently of their internal or external nature).
60. Cf. Lagrange (1764), pp. 7–8.
61. Cf. Lagrange (1780), pp. 221–3.
62. Cf. *ibid.*, p. 223. Thus, by beginning with the principle of virtual velocities, we can recognize the necessary condition to deduce the equations of motion from the principle of least action [cf. the note (35)]. We can probably find here one of the reasons for Lagrange's passage to the former principle [cf. Fraser (1983), p. 234]. [I am grateful for this and the note (35) to prof. Pulte].
63. Cf. the note (58) from which it is clear that

$$\left(\frac{\delta\alpha}{\delta d\varphi}\right)d\varphi+\left(\frac{\delta\alpha}{\delta d\psi}\right)d\psi+\&\text{c.}=\frac{\mathrm{M}}{dt^2}[d\mathrm{F}^2+d\mathrm{G}^2+d\mathrm{H}^2]=2\alpha$$

and so

$$\left(\frac{\delta T}{\delta d\varphi}\right)d\varphi+\left(\frac{\delta \mathrm{T}}{\delta d\psi}\right)d\psi+\&\text{c.}=2\mathrm{T}$$

64. Cf. Dahan-Dalmedico (1990), p. 94. I would not just say that "the *statut* of the calculus of variation

shifted", but I would rather talk of an elimination of this calculus as such, in spite of the central role played by δ-formalism [cf also *ibid.*, p. 101].

65. Cf. Lagrange (1788). I shall refer, here, to the first edition and mention the second [cf. Lagrange (1811–15)] only if relevant differences occur on the matter considered. According to the difference in the orthography of the titles of Lagrange (1788) and (1811–15) I shall refer to the former using the terms "*Méchanique analitique*" or "the first edition of the *Mécanique analytique*" and to the latter using the term "the second edition of the *Mécanique analytique*". I reserve the term "*Mécanique analytique*" for the reference to Lagrange's work as such and not specifically to its first or second edition. On Lagrange's *Mécanique analytique* cf., among the classic works: Mach (1883), pp. 458–71 and Dugas (1950), pp. 318–24.
66. Lagrange's words are explicit [cf. Lagrange (1788), *Avertissement*, p..V]:

    On a déjà plusieurs Traités de Méchanique, mais le plan de celuici est entiérement neuf. Je me suis proposé de réduire la théorie de cette Science, et l'art de résoudre les problêms qui s'y rapportent, à des formules générales dont le simple développement donne toutes les équations nécessaires pour la solution de chaque problême.

    According to Mach, "the mechanics of Langrange is a stupendous contribution to the economy of thought" [Cf. Mach (1883), p. 458].
67. Lagrange could write [cf. again Lagrange (1788), p. V.]:

    Cet Ouvrage aura d'ailleurs une autre utilité; il rénuira et présentera sous un même point de vue, les différens Principes trouvés jusqu'ici pour faciliter la solution des questions de Méchanique, en montrera la liason et la dépendance mutuelle, et mettra à portée de juger de leur justesse et de leur étendue.
68. To modify the motion of a body in a state of rest is to put it in motion.
69. Cf. *ibid.*, p. 1–2.
70. Cf. *ibid.*, p. 2.
71. Cf. *ibid.*, p. 12.
72. Cf. Lagrange (1811–15), t. I, p. 23.
73. Cf. *ibid.*, pp. 23–6. As both Jouguet and Dugas argues, Lagrange's demonstration is based on physical facts—on certain simple properties of pulleys and strings [Cf. Dugas (1950), p. 321 and Jouguet (1908–09), vol II, p. 179].
74. As in his 1760–61, 1764 and 1780 memoirs, in his *Méchanique analytique* also Lagrange denotes forces by P_i, Q_i, &c.. However in the latter he does not introduce masses in the general equation of principle of virtual velocities for *equilibrium* of a discrete system. In the *Statique* the meaning of the symbols P_i, Q_i, &c. have, then, to be taken as different than in the previous memoirs. However, to conform my language I shall continue to denote forces by M_iP_i, M_iQ_i, &c. [cf. the note (34)].
75. Cf. Lagrange (1815–15), t. I, p. 24.
76. Cf. *ibid.*, t. I, p. 26. The unmotivated passage from commensurable to incommensurable cases in a proof is a frequent topic in XVIIIth century mathematics. A very good example is Daniel Bernoulli's proof of decompositions of forces [cf. Bernoulli (1726), prop. I]. On this topic cf. Dhombres (1986–87).
77. Cf. Lagrange (1788), p. 44–5:

    Ceux qui jusqu'à présent ont écrit sur le Principe des vitesses virtuelles, se sont plutôt attachés à démontrer la vérité de ce principe par la conformité de ses résultats avec ceux des principes ordinaires de la Statique, qu'à monter l'usage qu'on en peut faire pour résoudre directement les problêmes de cette Science. Nous nous sommes proposé de remplir ce dernier objet avec toute la généralité dont il est susceptible et de déduire du Principe dont il s'agit, des formules analitiques qui renferment la solution de tous les problêmes sur l'équilibre des corps, à-peu-prés de la même maniere que les formules des soutangentes, des rayos osculateurs, &c., renferment la détermination de ces lignes dans toutes les courbes.
78. Cf. Lagrange (1788), p. 8. If a force causes modification of motion, without force a body moves

uniformly. Virtual velocity can be considered as the speed of this motion (initial speed).

79. Note that here Lagrange uses the differential symbol d to express these displacements and only in the fourth section [p. 51] does he introduce the difference between d and δ, recommending a replacement in the previous *formulæ*.

80. Cf. Lagrange (1788), p. 18.

81. We have, in fact,

$$\cos\alpha=\left(\frac{x-a}{p}\right),\quad \cos\beta=\left(\frac{y-b}{p}\right),\quad \cos\gamma=\left(\frac{z-c}{p}\right)$$

as we can verify with simple geometric considerations on right triangles.

82. If M_pP is external, the surface is a sphere of which the centre is the origin of force. If M_pP is internal, the surface is arbitrary. So by knowing the equation of the perpendicular surface to M_pP, we know, *a priori*, whether M_pP is internal or external [Cf. Lagrange (1811–15), t. I, pp. 39–43]. Cf. the next paragraph II. κ..

83. Cf. Lagrange (1811–15), t. I, pp. 39–43.

84. For this evident interpretation of (37) cf. Lagrange (1811–15), t. I, pp. 110–11. The condition between square brackets is due to Poinsot [cf. Poinsot (1846)]. It corresponds to the requirement that $\delta\varphi$, $\delta\psi$, &c. are the orthogonal projections of displacement of the point along the directions of forces expressed by Φ, Ψ, &c. [cf. Bertrand's note on page 39 of Lagrange (1853–55)]. To have the principle of levers is enough to combine the general equation of *equilibrium* with the equation of condition which expresses a system of levers.

85. This section is expanded and modified in the second edition.

86. As Dugas argues [Cf. Dugas (1950), p. 324]:

Sur la notion de *force*, Lagrange ne philosophe pas outre mesure.

87. Cf. Lagrange (1788), p. 158.

88. Cf. *ibid.*, p. 162. Lagrange's unclear passage is clarified in the second edition [Cf. Lagrange (1811–15), t. I, p. 224]. The infinitesimal attitude is not, of course, the only which can make this circular definition work. If we express a uniformly varied motion in space-time orthogonal coordinates, we have a curve such that the ratio between the rate of change of speed and the time is constant. If we give an algorithm to find instantaneous speed, we can prove that this curve is a parabola. Therefore the problem to measure the instantaneous effects of a force on a movable point whose motion is represented in space-time orthogonal coordinates by any curve, is to find the difference between space run in a given time by a first body whose motion is represented by the tangent to this curve, and a second body whose motion is represented by its osculatory parabola. Lagrange's central idea in the *Théorie* will be to found mechanics precisely on a non infinitesimal solution of this problem. For an analogous interpretation of the notion of instantaneous speed, cf. Maclaurin (1742), p. 53.

89. Thus we can simply prove that (37) and (38) express the usual law of composition of forces according to the parallelogram rule.

90. An interesting addition to the second edition is the explicit consideration of the general case of motion in a resistant *medium* [Cf. Lagrange (1811–15), t. I, pp. 252–3]. For this it is enough to introduce a new force -$M_pP\rho$ opposite to the direction of tangent (at curve described by motion) for each body. If the origin of this force is taken infinitely near to the body, we have (if a, b, c are the orthogonal coordinates of the origin):

$$\rho=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$$

$$\delta\rho=\frac{x-a}{\rho}\delta x+\frac{y-b}{d\rho}\delta y+\frac{z-c}{d\rho}\delta z=\frac{dx}{ds}\delta x+\frac{dy}{ds}\delta y+\frac{dz}{ds}\delta z$$

if the force is external (the *medium* is fixed); if, on the contrary, it is internal (the *medium* moves), we have:

$$\delta\rho = \frac{dx-da}{d\sigma}\delta x + \frac{dy-db}{d\sigma}\delta y + \frac{dz-dc}{d\sigma}\delta z$$

$$[d\sigma = \sqrt{(dx-da)^2+(dy-db)^2+(dz-dc)^2}]$$

91. Cf. respectively the previous paragraphs I. δ and I. θ..
92. Cf. Lagrange (1788) p. 199.
93. The centre of gravity is defined as the point such that the force of gravity is unable to communicate to the system any motion of rotation around it. Lagrange has proved however [cf. Lagrange (1788), pp. 35–6] that this point also has the property that

> la somme de chaque masse [multipliée] par sa distance à un plan passant par ce point, soit nulle relativement à trois plans perpendiculaires.

94. Cf. Lagrange (1788), p. 201. To explain the conclusion of Lagrange, let us consider the "translation" of this reasoning in the second edition where Lagrange explicitly introduces the distinction between internal and external forces. [Cf. (1811-15), t. I, pp. 257–60] Let $M_iX_i^e$, $M_iY_i^e$, $M_iZ_i^e$, and $M_iX_i^\wedge$, $M_iY_i^\wedge$, $M_iZ_i^\wedge$, be two systems of forces respectively parallel parallel to three orthogonal axes and equivalent (according to (37) and (38)) to external and internal forces occurring in (18) and let the system be completely free. Let $x_i = x+\xi_i$; $y_i = y+\eta_i$; $z_i = z+\zeta_i$ be the orthogonal coordinates (relative to the same axes) of the points of the system. By these simple replacements, the general equation (18) can be written in the following form:

$$\sum_{i=1}^{n} M_i \left[\begin{array}{l} \left(\dfrac{d^2x}{dt^2}+\dfrac{d^2\xi_i}{dt^2}\right)(\delta x+\delta\xi_i)+\left(\dfrac{d^2y}{dt^2}+\dfrac{d^2\eta_i}{dt^2}\right)(\delta y+\delta\eta_i)+ \\ \left(\dfrac{d^2z}{dt^2}+\dfrac{d^2\zeta_i}{dt^2}\right)(\delta z+\delta\zeta_i)+ \\ X_i^e(\delta x+\delta\xi_i)+Y_i^e(\delta y+\delta\eta_i)+Z_i^e(\delta z+\delta\zeta_i)+ \\ X_i^\wedge(\delta\xi_i)+Y_i^\wedge(\delta\eta_i)+Z_i^\wedge(\delta\zeta_i) \end{array}\right] = 0$$

which is a trivial consequence of the replacements, except for the terms relative to internal forces, the form of which depends on (34) [cf. *below*]. But, if point (x, y, z) is the centre of gravity of the system, according to (43) equating the coefficients of independent variations δx, δy and δz to zero, we directly have

$$\frac{d^2x}{dt^2}\sum_{i=1}^{n} M_i + \sum_{i=1}^{n} X_i^e M_i = 0$$

$$\frac{d^2y}{dt^2}\sum_{i=1}^{n} M_i + \sum_{i=1}^{n} Y_i^e M_i = 0$$

$$\frac{d^2z}{dt^2}\sum_{i=1}^{n} M_i + \sum_{i=1}^{n} Z_i^e M_i = 0$$

which replaces (44). The absence of internal forces expresses the fact that this motion is independent of these forces and, thus, of mutual actions of the bodies in the system, i.e. precisely the content of the principle of conservation of movement of the centre of gravity. This principle is also a simple (weaker) consequence of the general equation of motion of the centre of gravity that analytically derives from the general *formula* of dynamics which is, in fact, stronger than this principle.

To justify the absence of variations δx, δy and δz from the terms relative to internal forces in the previous transformed form of (18) let us consider $M_iP_i^\wedge$, $M_iQ_i^\wedge$, &c. as the internal forces of the system. From (34) we shall have, for every $i (i = 1, 2, \ldots, n)$:

$$\delta p_i^\wedge = \left(\frac{x_i - x_{v_i}}{p_i^\wedge}\right)(\delta x_i - \delta x_{v_i}) + \left(\frac{y_i - y_{v_i}}{p_i^\wedge}\right)(\delta y_i - \delta y_{v_i}) + \left(\frac{z_i - z_{v_i}}{p_i^\wedge}\right)(\delta z_i - \delta z_{v_i})$$

$$\delta q_i^\wedge = \left(\frac{x_i - x_{\mu_i}}{q_i^\wedge}\right)(\delta x_i - \delta x_{\mu_i}) + \left(\frac{y_i - y_{\mu_i}}{q_i^\wedge}\right)(\delta y_i - \delta y_{\mu_i}) + \left(\frac{z_i - z_{\mu_i}}{q_i^\wedge}\right)(\delta z_i - \delta z_{\mu_i})$$

&c.

where $(x_{v_i}, y_{v_i}, z_{v_i})$, $(x_{\mu_i}, y_{\mu_i}, z_{\mu_i})$, &c $(1 \leq v_i, \mu_i, \&c \leq n)$ are the points of the system which are the origins of forces $\mathrm{M}_i\mathrm{P}_i^\wedge$, $\mathrm{M}_i\mathrm{Q}_i^\wedge$, &c.. Therefore the coefficients of δx_i are:

$$\mathrm{X}_i^\wedge = \left(\frac{x_i - x_{v_i}}{p_i^\wedge}\right) + \left(\frac{x_i - x_{\mu_i}}{q_i^\wedge}\right) + \&c.$$

while the coefficients of δx_{v_i}, δx_{μ_i}, &c. are:

$$\mathrm{X}_{v_i}^\wedge = \&c. - \left(\frac{x_i - x_{v_i}}{p_i^\wedge}\right) - \&c.$$

$$\mathrm{X}_{\mu_i}^\wedge = \&c. - \left(\frac{x_i - x_{\mu_i}}{q_i^\wedge}\right) - \&c.$$

&c.

Thus, substituting the respective values of x_i, and of x_{v_i}, x_{μ_i}, &c. we shall have (because of v_i, μ_i, &c. $(i = 1, 2, \ldots, n)$ are values also taken by index i in its variation from 1 to n):

$$\mathrm{X}_i^\wedge(\delta x) = \left(\frac{\xi_i - \xi_{v_i}}{p_i^\wedge}\right)\delta x + \left(\frac{\xi_i - \xi_{\mu i}}{q_i^\wedge}\right)\delta x + \&c.\,; \quad \mathrm{X}_i^\wedge(\delta \xi_i) = \left(\frac{\xi_i - \xi_{v_i}}{p_i^\wedge}\right)\delta\xi_i + \left(\frac{\xi_i - \xi_{\mu_i}}{q_i^\wedge}\right)\delta\xi_i + \&c.$$

$$\mathrm{X}_{v_i}^\wedge(\delta x) = \&c. - \left(\frac{\xi_i - \xi_{v_i}}{p_i^\wedge}\right)\delta x - \&c.\,; \quad \mathrm{X}_{v_i}^\wedge(\delta \xi_{v_i}) = \&c. - \left(\frac{\xi_i - \xi_{v_i}}{p_i^\wedge}\right)\delta\xi_{v_i} - \&c.$$

$$\mathrm{X}_{\mu_i}^\wedge(\delta x) = \&c. - \left(\frac{\xi_i - \xi_{\mu_i}}{q_i^\wedge}\right)\delta x - \&c.\,; \quad \mathrm{X}_{\mu_i}^\wedge(\delta \xi_{\mu_i}) = \&c. - \left(\frac{\xi_i - \xi_{\mu_i}}{q_i^\wedge}\right)\delta\xi_{\mu_i} - \&c.$$

and so, in general (again because of v_i, μ_i, &c. $(i = 1, 2, \ldots, n)$ are values also taken by index i in its variation from 1 to n):

$$\sum_{i=1}^{n} \mathrm{X}_i^\wedge \delta x = 0$$

and the same for the other variables.

95. Here, I follow directly the second edition [cf. Lagrange (1811–15), t, I, pp. 262–65]. The difference between the two editions is, in fact much less relevant here than in the previous case [cf. Lagrange (1788), p. 202–05].

96. If $\mathrm{M}_i\mathrm{P}_i^\wedge$ $(i = 1, 2, \ldots, n)$ are internal forces whose origins are points of orthogonal coordinates $(x_{v_i}, y_{v_i}, z_{v_i})$ according to (34), we have:

$$\begin{aligned} p_i^\wedge \delta p_i^\wedge &= (x_i - x_{v_i})(z_i\delta\iota - y_i\delta\theta - z_{v_i}\delta\iota + y_{v_i}\delta\theta) \\ &\quad + (y_i - y_{v_i})(x_i\delta\theta - z_i\delta\kappa - x_{v_i}\delta\theta + z_{v_i}\delta\kappa) \\ &\quad + (z_i - z_{v_i})(y_i\delta\kappa - x_i\delta\iota - y_{v_i}\delta\kappa + z_{v_i}\delta\kappa) = \\ &= [0]\delta\theta + [0]\delta\iota + [0]\delta\kappa = 0 \end{aligned}$$

97. If the origin of coordinates is also the common origin of all the external forces, all the distances

p_i^e, q_i^e, &c. are equal the one to each other and correspond to the *radius* vector K_i of the points (x_i, y_i, z_i). Thus, from (38) we shall have:

$$X_i^e = (P_i^e + Q_i^e + \&c.)\left(\frac{x_i}{K_i}\right)$$

$$Y_i^e = (P_i^e + Q_i^e + \&c.)\left(\frac{y_i}{K_i}\right)$$

$$Z_i^e = (P_i^e + Q_i^e + \&c.)\left(\frac{z_i}{K_i}\right)$$

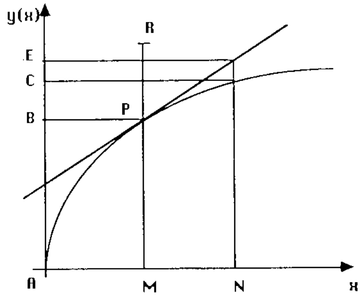
and so:

$$Y_i^e x_i = X_i^e y_i\,; \quad X_i^e z_i = Z_i^e x_i\,; \quad Z_i^e y_i = Y_i^e z_i$$

98. According to Lagrange's proof in the 1780 memoir [cf. the previous paragraph I.θ.] this condition is always satisfied. Lagrange presents a similar proof also in the fourth section of the second part of the *Méchanique analitique* [cf. Lagrange (1788) p. 225 and following]. He writes, relatively to the transformation of $P\delta p + Q\delta q + \&c.$ in a function of φ, ψ, ω, &c. [cf. *ibid.*, p. 225]:

   cette operation dévient encore plus facile, lorsque les forces sont telles que la somme des moments, c'est-à-dire la quantité $Pdp + Qdq + Rdr + \&c.$ est intégrable, ce qui [...] est proprement le cas de la nature.

99. Cf. Lagrange (1788), pp. 206–07.
100. Cf. the note (22).
101. Cf. Lagrange (1788), p. 211.
102. Let us consider a function $y = y(x)$ and let $AM = x$ and $MP = y(x)$. We can consider increment of $y(x)$ in different ways.



i) We can increase x by dx and consider the increment of $y(x)$ equal to:

$$y(x+dx) - y(x) = y'(x)dx + \frac{dx^2}{2!}y''(x) + \&c. = BC$$

that is the total difference of $y = y(x)$.

ii) We can increase x by dx and consider the increment of y(x) equal to:

$$dy = d[y(x)] = y'(x)dx = BE$$

that is the differential of $y = y(x)$.

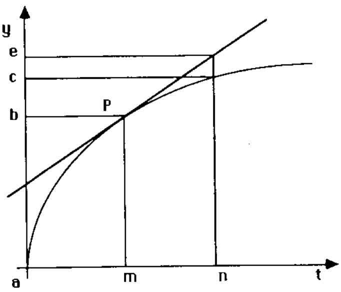
iii) We can increase y directly by an arbitrary and independent increment

$$\delta y = PR$$

that is the variation.

The difference between (ii) and (iii) is, clearly, that in (ii) y is taken as a generic function $y=y(x)$ $[y'\neq 1]$, while in (iii) it is taken as an independent variable $[y'=1]$.

Now, let x and y be two fonctions of a common variable t. Let also be $y=f(x)$. We can consider variations (which are increments independent of f, but not of the functional link between y and t) almost in two different ways ($mn=\delta t$; $t'=1$):



iv) $\delta[y(t)]=y'(t)\delta t+y''(t)\frac{\delta t^2}{2!}+\&c.=bc$

v) $\delta[y(t)]=y'(t)\delta t=be$

If we consider the variation only as dependent on the punctual situation, we have to choose (v).

The difference between (ii), (iii) and (v) lies in the choice of independent variable. In case (ii) we have chosen $x[x'=1]$; in case (iii) we have chosen $y[y'=1]$, in case (v) we have chosen $t[t'=1]$.

Virtual velocities of mass-point (x, y, z) along the directions of the axes x, y, z are, then, expressed by the variations δx, δy, δz $[x'=1,\ y'=1,\ z'=1]$, while virtual velocity of the same mass-point along direction p is expressed by:

$$\delta p=\delta[p(x, y, z)]=\delta_x p\delta x+\delta_y p\delta y+\delta_z p\delta z$$

which is the first increment of p relative to increments δx, δy and δz of x, y and z. Note that δp is not virtual velocity of p (a line has no virtual velocity), but the changing of p which corresponds to changing of position of body produced by its virtual velocity).

This reasoning uses only very Lagrangian tools and we shall see that the general idea of Lagrange in the *Théorie* does not differ deeply from it.

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The Analytical Foundation of Mechanics of Discrete Systems in Lagrange's *Théorie des Fonctions Analytiques*, Compared with Lagrange's Earlier Treatments of This Topic. Part 2

Marco PANZA*

II. The Foundation of Mechanics of Discrete Systems in the *Théorie des fonctions analytiques*.

α. Although Lagrange wrote his *Théorie des fonctions analytiques*[103] for didactic reasons,[104] it cannot be considered a simple text-book. The basic justification for Lagrange's work is epistemological and its goal is to argue for a philosophical thesis: all mathematics can be reduced to algebraic analysis of finite quantities.[105] To show how to implement this project, Lagrange provides a functional, non differential interpretation of *calculus* (what he calls theory of analytical functions) and outlines its "applications" to geometry and mechanics of discrete systems.

A widespread argument against applicability of Lagrange's theory both in mathematical research and in teaching was difficulty in using it to solve current specific problems and to explain natural phenomena.

The following remark by de Prony is a good example:

> Cette théorie est assurément une très-intéressante partie de ce qu'on pourrait appeller l'étude purement *philosophique* des mathématiques; mais quand il s'agit de faire de l'analyse transcendante un *instrument* d' exploration pour les questions que présentent l'astronomie, la marine, la géodesie et les différentes branches de la science de l'ingénieur, la considération des infiniment petits conduit au but d'une manière plus facile, plus prompte, plus immédiatement adaptée à la nature de ces questions, et voilà pourquoi la mèthode *leibnitienne* a, en général, prévalu dans les écoles francaises.[106]

Lazare Carnot was, if possible, even more unequivocal:

> Le véritable obstacle à l'adoption d'une méthode aussi lumineuse, est la nouveauté de l'algorithme, pour lequel il faudrait abandonner celui qu'un longue habitude a consacré, et, d'après lequel sont rédigés tous les ouvrages originaux qui ont paru depuis un siècle; ainsi, par exemple, il faudrait refondre

* Univ. de Genève, Enseignement d'Histoire et Philosophie des Sciences

> toutes les collections académiques, tous les écrits d'Euler et ceux de Lagrange lui-même. Cette pensée était la sienne, lorsqu'il publia la nouvelle édition de sa *Mécanique analytique*; il n'y emploie point son algorithme [...].[107]

To support his judgement, Carnot quotes the following passage of Lagrange's *avertissement* of the second edition of his treatise:

> On a conservé la notation ordinaire du Calcul différentiel, parce qu'elle répond au système des infiniment petits, adopté dans ce Traité. lorsqu'on a bien conçu l'esprit de ce système, et qu'on s'est convaincu de l'exactitude de ses résultats par la méthode géométrique des premières et dernières raisons, ou par la méthode analytique des fonctions dérivées, on peut employer les infiniment petits comme un instrument sûr et commode pour abréger et simplifier les démonstrations.[108]

The historical judgement which has often been drawn on the basis of similar remarks, is the existence of a mathematical and philosophical opposition between Lagrange's two main treatises. The origins of this judgement lie in the historical and mathematical overstimate of the use of infinitesimal quantities and notations as a conceptual watershed in the history of *calculus*.

In spite of this general conviction, if we consider the mathematical framework of Lagrange's use of infinitesimals both in mechanics, and in his previous analytical works, we can reconstruct a consistent path of which the *Théorie* is a natural outcome. The permanence of differential notations in the second edition of the *Mécanique Analytique* marks a very strict correspondence between the mathematical organization of mechanics in *Mécanique analytique* and in *Théorie des fonctions analytiques*. De Prony's and Carnot's remarks are rather the sign of a widespread incomprehension of Lagrange's approach.

The point of *Mécanique analytique* is—as Lagrange writes both in the first and the second edition—to make mechanics "une nouvelle branche" of analysis.[109] The point of the *Théorie* is to reduce all mathematical analysis to algebraic analysis. Thus it is natural that in the latter work, Lagrange outlines consequences of his reduction in every branch of analysis and that in the second edition of the former, he does not return to the question. The two works are *different* parts of the same programme.

β. The whole analytical structure of the *Théorie des fonctions analytiques* is founded on the separation of formal context from numerical context. This separation is conceived as legitimate on the presumption that for every function $f(x)$ there exists one—and only *one*—development of type $f(x)+p\xi^{\alpha}+q\xi^{\beta}+$&c. ($\alpha$, β, &c. rationals and $p=p(x)$, $q=q(x)$, &c.) such that it converges to $f(x+\xi)$ for a sufficient small ξ. Lagrange proves that "in general" (i.e.: for every x except for isolated values), α, β, &c. are natural numbers. Thus it is always possible to develop $f(x+\xi)$ in a power series converging to such a funation if ξ is sufficiently small.[110]

On this basis, we can conceive a general theory of functions as a theory of developments in power series and found its numerical applications on general results about evaluation of the remainder.

In particular, if we denote p with $f'(x)$, q with $f''(x)/2!$ &c., Lagrange proves that:[111]

i) ξ can even be sufficiently small that for every natural n

$$\frac{f^{(n)}(x)}{n!}\xi^n > \sum_{k=n+1}^{\infty} \frac{f(x)}{k!}\xi^n$$

ii) the remainder of the series can be written in the form

$$\frac{f^{(n+1)}(x+v)}{(n+1)!}\xi^{n+1}$$

where $0<v<\xi$, so that, "in general", we have:

$$f(x+\xi)=f(x)+f'(x)\xi+\frac{f''(x)}{2!}\xi^2+\ \dots\ +\frac{f^{(n)}(x)}{n!}\xi^n+\frac{f^{(n+1)}(x+\lambda\xi)}{(n+1)!}\xi^{n+1} \tag{1}$$

$$[0\leq\lambda\leq 1]$$

After having proved (1), Lagrange writes:

> La perfection des méthodes d'approximation dans lesquelles on emploie les séries, dépend non-seulement de la convergence des séries, mais encore de ce qu'on puisse estimer l'erreur qui resulte des termes qu'on néglige, et à cet égard on peut dire que presque toutes les méthodes d'approximation dont on fait usage dans la solution des problèmes géometriques et mécaniques, sont encore très imparfaites. Le théorème précédent pourra servir dans becaucoup d'occasion à donner à ces méthodes la perfection qui leur manque, et sans laquelle il est souvent dangereux de les employer.[112]

By "méthodes d'approximation" Lagrange means the use of truncated series to represent a function.

In this sense, in the *Théorie* the entire mechanics of discrete system is founded on a rigourous, "perfected" method of "approximation". If we represent the space crossed by a moving body in time t by function $s=f(t)$, Lagrange shows that, to determine the equation of motion of this body in instant t, we consider the difference $f(t-\vartheta)-f(t)$—expressing the space run in time ϑ—as perfectly represented by the truncated series $f'(t)\vartheta+(\vartheta^2/2!)f''(t)$ (in Lagrange's language: "on peut faire abstraction des autres termes"[113]).

γ. According to Lagrange, motion is represented, by a *functional* relation between space and time. If we consider space as represented in a system of three orthogonal coordinates x, y, z, we have, in general, $x=x(t)$, $y=y(t)$, $z=z(t)$. Thus mechanics can be considered as a "géométrie à quatre dimensions",[114] and an analytical interpretation of geometry is the basis of the reduction of mechanics to analysis.

Therefore, if we are able to justify the representation of velocity of an unitary mass-point[115] by $f'(t)$ and of force acting upon it by $f''(t)$ (being its motion generally represented by $f(t)$), all *formulæ* of the *Méchanique analitique* can be repeated with a simple linguistic replacement. But in the new context, Lagrange has no recourse to the principle of virtual velocities, to justify them. A strictly mathematical deduction starting only from the Newtonian laws is necessary to realise his programme.[116] In the last paragraph of part I, I have shown that it is possible. Lagrange's path is not very different.

δ. Let us start with Lagrange's justification of derivative representation of speed and force. If motion is generally represented by a functional relation expressing space covered in a given time, we can always represent the motion along time $t+\vartheta-t=\vartheta$, by the functional difference:

$$f(t+\vartheta)-f(t)=f'(t)\vartheta+\frac{f''(t)}{2!}\vartheta^2+\frac{f'''(t)}{3!}\vartheta^3+\&c. \tag{2}$$

i.e.: all motions (spaces) can be considered as composed (as a sum) by motions (spaces) of type $a\vartheta$, $b\vartheta^2$, $c\vartheta^3$, &c..

Let us consider space represented by a straight line segment x. The function $x=a\vartheta$ represents a rectilinear motion where the spaces are always proportional to time. A similar motion is called *uniform* and the constant a—expressing constant ratio between space and time—is proportional to its speed, that is constant. *By experience* we know this is the motion of a moving body "si on écarte toutes les causes d'altération qui peuvent agir sur lui". Thus, we have, *by experience*, the first law of motion:

> la vitesse une fois imprimée, se conserve toujours la même et suivant la même direction.[117]

The function $x=b\vartheta^2$ represents, on the contrary, a rectilinear motion, where spaces are always proportional to square of time. If the ratio between space and time expresses speed, b is the ratio between speed and time, which is here constant. Because the rate of change of space is increasing, we call this motion accelerated; it will be produced by a force acting continuously upon the body, and acceleration will be measured by the ratio between speed and time. Constant b will measure the accelerative force and motion will be *uniformly accelerated. By experience* we know this is the motion of falling bodies "en faisant abstraction de la résistence de l'aire, et de toute autre cause étrangère d'altération".[118]

The functions $c\vartheta^3$, &c. do not represent any natural motion and

> nous ignorons ce que le coefficient c pourrait réprésenter, en le considérant d'une manière absolue et indépendante des vitesses ed des forces.[119]

By composing the two first functions, we have a composed motion, represented by $x=a\vartheta+b\vartheta^2$, which could be considered composed (as a sum) of a uniform and a uniformly accelerated motion. By adding rectilinear and co-directed motions as straight segments, we may consider this motion as resulting from a constant speed proportional to a, and by an accelerative force (changing the effective speed of the body along time ϑ) proportional to b.

Here, a and b are considered constant measures of mean speed and acceleration of the first and the second motion respectively. If we want to measure instantaneous speed and acceleration of composed motion, we have to express mean speed and acceleration of a moving body that moves with the same speed and the same acceleration of the selected instant along all ϑ. But if the selected instant is the initial one of time ϑ, a is proportional to the effective instantaneous speed (not changed again by accelerative force) and b is proportional to effective accelerative force instantaneously acting upon the body, as it is constant along time ϑ. So, at this initial instant of time ϑ, a and b are really proportional to the instantaneous speed and acceleration of the rectilinear motion represented by the function $x = a\vartheta + b\vartheta^2$.

The initial instant of time ϑ is, of course, the instant expressed by t in (2), thus, if we prove that for every function $x = a\vartheta + b\vartheta^2$, we can take a sufficient small value of ϑ for the absolute value of difference between $f(t+\vartheta) - f(t)$ (for very f) and $\vartheta f'(t) + (\vartheta^2/2)f''(t)$ will be lower than the absolute value of difference between $f(t+\vartheta) - f(t)$ and $a\vartheta + b\vartheta^2$, we shall have proved that

> on peut prendre ϑ assez petit pour que le mouvement composé des deux termes $\vartheta f'(t) + (\vartheta^2/2)f''(t)$ approche plus du veritable mouvement que ne pourrait faire tout autre mouvement composé d'un mouvement uniforme et d'un mouvement uniformément accéléré [i.e. any motion $x = a\vartheta + b\vartheta^2$].[120]

Therefore, for every motion $x = f(t)$, the instantaneous speed in t will be proportional to the mean speed along ϑ of motion $x = f'(t)\vartheta$, that is $f'(t)$, and instantaneous acceleration in t will be proportional to mean acceleration along ϑ of motion $x = (\vartheta^2/2)f''(t)$, i.e.: $f''(t)/2$. Thus we can generally measure velocity and acceleration in instant t of any motion $x = f(t)$ respectively with $f'(t)$ and $f''(t)$ (that is proportional to $f''(t)/2$).

Even if the measure of a speed can be directly intended as a representation of it, this is not the case for a force, that, depending on the mass of the moving body, is rather represented by a quantity proportional to its measure. However if mass is taken as unitary we can represent force by its measure too. Therefore, to justify the previous representations in terms of derivative functions of speed and force, we have only to prove, according to (1), that:

(*) for every a and b, we can take a sufficient small value of ϑ, s.t.

$$(3)\qquad \left|\frac{f'''(t+\lambda\vartheta)}{3!}\vartheta^3\right| < \left|[f'(t)-a]\vartheta + \left[\frac{f''(t)}{2!} - b\right]\vartheta^2 + \frac{f'''(t+\lambda\vartheta)}{3!}\vartheta^3\right| \qquad [0 \le \lambda \le 1]$$

Lagrange states his result without explicit reference to absolute values and inverting the order of universal and existential quantification ("we can take a small enough value of ϑ s.t., for every motion of type $x = a\vartheta + b\vartheta^2, \ldots$"). However, the proof clearly shows that the demonstrated proposition is, in fact, (*), which is really needed. Inversion of quantifiers (or analogous misunderstandings) are common in mathematical works before the middle of the XIXth-century. The logical difference between the two formulations (and concepts) was not understood. In the best of cases—as in this one—where only a weaker formulation is needed, it simply involves a certain linguistic inexactness; whereas in other cases, it involves very important mathematical mistakes.

However, in this corrected form, the theorem is a particular case of a general geometric results, proved by Lagrange in the second part of the *Théorie*[121] (applications to geometry): no curve expressed by an ordinate of which even one of the first n derivatives is different by respectively $f'(t_0), f''(t_0), \ldots, f^{(n)}(t_0)$ can pass—around point t_0—between the curves $x=f(t)$ and $x=f(t_0)+tf'(t_0)+\ \ldots\ +(t^n/n!)f^{(n)}(t_0)$; i.e.; $x=f(t_0)+tf'(t_0)+\ \ldots\ +(t^n/n!)f^{(n)}(t_0)$ is n-degree osculatory curve of $x=f(t)$ at $t=t_0$.

Instead of recalling this general result and considering (*) as a simple corollary, Lagrange recalls its proof, asserting that (*) can be proved by "un raisonnement semblable".[122] The reason for this strange procedure is probably that in the recalled proof, derivatives occur only as graphic symbols for coefficients. So (*) can be proved without any expressed reference to the notion of derivative functions.

Lagrange's argument is, however, extremely obscure if we take it literally. By introducing explicit reference to absolute values and adapting the argument, we can reconstruct it in the following terms, assuming that $f'(t), f''(t)$ and $f'''(t)$ are continuous functions (around point t), which Lagrange takes for granted.[123]

Let us note the coefficient of ϑ, ϑ^2 and ϑ^3 in (3) by A, B, C. If $\mathrm{A}\neq 0$, it is evident that we can always take a positive number η_1 such that if $0<\vartheta\leq\eta_1$, the trinomial $\mathrm{A}+\mathrm{B}\vartheta+\mathrm{C}\vartheta^2$ has the same sign as A. Thus:

(4) i) if $\mathrm{A}>0$ and $0<\vartheta\leq\eta_1$, then $\mathrm{A}+\mathrm{B}\vartheta+\mathrm{C}\vartheta^2>0$

ii) if $\mathrm{A}<0$ and $0<\vartheta\leq\eta_1$, then $\mathrm{A}+\mathrm{B}\vartheta+\mathrm{C}\vartheta^2<0$.

Taking $\mathrm{A}>0$, and dividing by ϑ (being always $\vartheta>0$), the inequality (3) becomes:

(5) $$\mathrm{A}+\mathrm{B}\vartheta+[\mathrm{C}-|\mathrm{C}|]\vartheta^2>0$$

and we can always take a positive number η_2 such that if $0<\vartheta\leq\eta_2$, then (5) is true (the trinomial has the same sign as A).

If $\mathrm{A}<0$, the inequality becomes:

(6) $$\mathrm{A}+\mathrm{B}\vartheta+[\mathrm{C}-|\mathrm{C}|]\vartheta^2<0$$

which is true if $0<\vartheta\leq\eta_2$, as before.

If $\mathrm{A}=0$ and $\mathrm{B}\neq 0$, we can always take a positve number η_3, such that the binomial $\mathrm{B}+\mathrm{C}\vartheta$ has the same sign as B and the proof runs analogously.

If $\mathrm{A}=0$ and $\mathrm{B}=0$, the theorem is meaningless.

If Lagrange could not formulate a similar proof, the essential point is that (*) can be proved in strictly algebraic terms, since it is an algebraic proposition.

Lagrange's commentary is the following:

> On peut conclure de là que tout mouvement rectiligne représenté par l'équation $x=f(t)$, peut, dans un instant quelconque au but du temps t, être regardé comme composé d'un mouvement uniforme dû à une vitesse imprimée au mobile, mesurée par $f'(t)$, et d'un mouvement uniformement accéléré dû

à une force accélératrice agissant sur le mobile et proportionelle à $(1/2)f''(t)$, ou simplement à $f''(t)$; que, par conséquent, si les causes qui empêchent le mouvement proposé d'être uniforme, venaient à cesser tout-à-coup, le mouvement se continuerait, dès cet instant, d'une manière uniforme avec une vîtesse mesurée par $f'(t)$; et que si l'effet de ces causes au lieu de devenir nul, devenait constant, le mouvement deviendrait composé du mouvement uniforme dont nous venous de parler, et d'un mouvement uniformément accéléré, commençant au même instant, en vertu d'une force accélératrice constant et proportionelle à $f''(t)$[...].

Donc, en général, dans tout mouvement rectiligne dans lequel l'espace parcouru est une fonction donnée du temps écoulé, la fonction prime de cette fonction représentera la vîtesse, et la fonction seconde représentera la force accéleratrice dans un instant quelconque[...].[124]

ε. Lagrange's justification of the representation of speed and force by derivatives is absolutely independent of algorithm of derivative functions. Speed and force are simply respectively measured (and represented) by the first and the second coefficient of Taylor's development of difference $f(t+\vartheta)-f(t)$. The search and the possession of a standard algorithm to determine these coefficients is an independent question.

With these first pages of the mechanical part of *Théorie des fonctions analytiques* Lagrange seems to realize a completely analytical version of the fluxional programme (particularly in Maclaurin's interpretation).

The central idea of this programme[125] was that the instantaneous speed of a moving body (its fluxion) was *measured* by the *mean* speed of a body moving along a given (not necessarily infinitesimal) time by a uniform motion with the same speed as the first body has in that selected instant. Analogously, the instantaneous acceleration of the same body is *measured* by the *mean* acceleration of another body moving along the same given time by a uniformly accelerated motion with the same acceleration as the first body in the selected instant. Thus, in this perspective, the central foundational problem was to determine the virtual motions with constant speed and acceleration to associate with every motion. If geometric representation by tangent and osculatory parabola (to the curve representing the effective motion) is not problematic, the general determination of analytical form of the functions representing these motions is always dependent, before Lagrange, on differential algorithm (however justified). Langrange is the first to translate the geometric definition of the tangent and the osculatory parabola into a general analytical form, independent of any particular algorithm. Differential algorithm may be looked for only subsequently. The foundation of *calculus* and mechanics is independent of it. The central proof to legitimate this foundation (granted the general form of power development) is very algebraic.

To achieve his result, Lagrange follows path opposed to the classical one. Instead of looking for uniform and uniformly accelerated motion with the appropriate mean speed and acceleration (or the straight line and the parabola respectively with the same direction and curvature as the original curve), he considers every motion as compounded and analyses the mechanical properties of the component motions, finding that the first component motion is uniform with a mean (constant) speed equal to the instantaneous speed of the original motion at the initial instant of the time and the second is uniformly accelerated, with a mean (constant) acceleration equal to instantaneous acceleration of the original motion at the same instant. So, to study the instantaneous speed and acceleration of every motion, we can limit ourselves to studying the first and second of its component motions. The reduction to simpler elements is the methodological tool of Lagrange's deduction.

Starting from these results, Lagrange's programme was to build a general geometry of speeds and forces.

If the algorithmic expression of Taylor's coefficient is given, the general mechanical *formulæ* can take a numerical content, i.e.: we can use them to actually calculate the motion of bodies. If this expression is not given, general mechanical *formulæ* remain the general expression of formal relations between mechanical entities and Taylor's coefficients.

The distinction between form and value is complete. We can express the form of a mechanical entity, even if we cannot express its value.[126]

So, if the internal organization of mechanics, after having provided a new functional representation of speed and force, can remain substantially the same as in *Méchanique analitique*, the mathematical meaning of *formulæ* changes completly. If we want to emphasize the internal organization of mechanics and provide a simple list of general *formulæ* to effectively solve mechanical problems, the classic differential approach is completely analogous to the derivative one. Thus, there was no reason to change proof and notations in the second edition of the *Mécanique analytique*. On the other hand, if we want to emphasize the genetic foundation of all mechanics on algebraic operations and affirm the general idea of separation between form and value, a derivative reformation of mechanics is essential.

ζ. Let τ a temporal variable and t a typical instant (a typical value of τ) and let us consider three rectilinear motions $x=x(\tau)$, $y=y(\tau)$ and $z=z(\tau)$ along directions of three orthogonal axes x, y, z. The curvilinear motion of a body can be expressed by the trajectory of a generic point $\mathrm{T}(x(t), y(t), z(t))$ in coordinates system x, y, z. Thus, the rectilinear motions $x=x(\tau)$, $y=y(\tau)$ and $z=z(\tau)$ are the projections on the axes x, y, z of this curvilinear motion.

In the instant t, the body will be at point T and if we want to study its instantaneous speed, we can limit ourselves to composing the virtual rectilinear motions with a mean speed equal to the instantaneous speed of the body in instant t (i.e.: in point T). If we consider as a parameter a given time ϑ, these motions are expressed by spaces:

$$\begin{aligned} \mathrm{X}(\vartheta) &= x(t)+x'(t)\vartheta-x(t)=x'(t)\vartheta \\ \mathrm{Y}(\vartheta) &= y(t)+y'(t)\vartheta-y(t)=y'(t)\vartheta \\ \mathrm{Z}(\vartheta) &= z(t)+z'(t)\vartheta-z(t)=z'(t)\vartheta \end{aligned} \tag{7}$$

If we want, on the contrary, to study the instantaneous acceleration, we can limit ourselves to composing virtual rectilinear motions with a mean acceleration equal to the instantaneous acceleration of the body in instant t. For the same time parameter ϑ, these motions are expressed by spaces[127]:

$$X(\vartheta)=x(t)+(1/2)x''(t)\vartheta^2-x(t)=(1/2)x''(t)\vartheta^2 \tag{8}$$
$$Y(\vartheta)=y(t)+(1/2)y''(t)\vartheta^2-y(t)=(1/2)y''(t)\vartheta^2$$
$$Z(\vartheta)=z(t)+(1/2)z''(t)\vartheta^2-z(t)=(1/2)z''(t)\vartheta^2$$

The general problem of determining the equations of motion of a moving body is, therefore, translated into the problem of determining geometric relations between the coefficients in (7) and (8) and speeds or forces (which is the problem of determining the composition laws of speeds and forces).

Let us start with speed. By eliminating ϑ in the three equations (7) and choosing $\mathrm{X}(\vartheta)$ as relative independent variable, we have:

$$\mathrm{Y}(\vartheta)=\mathrm{X}(\vartheta)\frac{y'(t)}{x'(t)};\quad \mathrm{Z}(\vartheta)=\mathrm{X}(\vartheta)\frac{z'(t)}{x'(t)} \tag{9}$$

which, if $\mathrm{X}(\vartheta)$, $\mathrm{Y}(\vartheta)$ and $\mathrm{Z}(\vartheta)$ are taken as variables, express a straight line drawn in a three dimensional space passing through point T and tangent to the curvilinear trajectory of the body. Thus the rectilinear space described (along the direction of this line) in time ϑ with a constant speed equal to the speed of the moving body at point T (at t), will be the rectilinear distance between the points $\mathrm{T}(x(t), y(t), z(t))\equiv \mathrm{T}(\mathrm{X}(0), \mathrm{Y}(0), \mathrm{Z}(0))$ and $(\mathrm{X}(\vartheta), \mathrm{Y}(\vartheta), \mathrm{Z}(\vartheta))$, i.e.:

$$\sqrt{\mathrm{X}(\vartheta)^2+\mathrm{Y}(\vartheta)^2+\mathrm{Z}(\vartheta)^2}=\vartheta\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2} \tag{10}$$

But this space can represent a uniform motion, the constant speed of which measures the instantaneous speed of the moving body at t. Thus, by noting this speed with $v(t)$, we have:

$$v(t)=\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}=s'(t) \tag{11}$$

where $x'(t)$, $y'(t)$ and $z'(t)$ express the instantaneous speed of component motions $x=x(\tau)$, $y=y(\tau)$ and $z=z(\tau)$ and $s(\tau)$ is the curvilinear space covered by the body in time τ.

Now, since $x'(t)\vartheta$, $y'(t)\vartheta$, $z'(t)\vartheta$ are the projections of right segment $\vartheta v(t)$ on the axes x, y, z, if we note with ε, ζ, η the angles formed by the right line expressed by the equations (9) and these axes, the law of decomposition of speed along three orthogonal directions is simply expressed by the equations:

$$x'(t)=v(t)\cos\varepsilon;\; y'(t)=v(t)\cos\zeta;\; z'(t)=v(t)\cos\eta \tag{12}$$

If the effective motion of a body is the result of composition of m different motions[128] with instantaneous speed in t equal to $v_j(t)=s_j'(t)$ $(j=1, 2, \ldots, m)$, by composing speed along commune directions as rectilinear spaces and by denoting by $v(t)=s'(t)$ the instantaneous effective total speed of the body we have:

$$v(t)=\sqrt{\left[\sum_{j=1}^{m} x_j'(t)\right]^2+\left[\sum_{j=1}^{m} y_j'(t)\right]^2+\left[\sum_{j=1}^{m} z_j'(t)\right]^2} \tag{13}$$

Thus, if ε, ζ, η and ε_j, ζ_j, η_j are respectively the angles of direction of $v(t)$ and $v_j(t)$ $(j=1, 2, \ldots, m)$ with axes x, y, z and we denote by $x'(t)$, $y'(t)$ and $z'(t)$ the orthogonal component of the effective total speed, from (12) we deduce directly:

$$\begin{aligned} v(t)\cos\varepsilon=s'(t)\cos\varepsilon&=\sum_{j=1}^{m} x_j'(t)=\sum_{j=1}^{m} v_j(t)\cos\varepsilon_j=\sum_{j=1}^{m} s_j'(t)\cos\varepsilon_j=x'(t)\\ v(t)\cos\zeta=s'(t)\cos\zeta&=\sum_{j=1}^{m} y_j'(t)=\sum_{j=1}^{m} v_j(t)\cos\zeta_j=\sum_{j=1}^{m} s_j'(t)\cos\zeta_j=y'(t)\\ v(t)\cos\eta=s'(t)\cos\eta&=\sum_{j=1}^{m} z_j'(t)=\sum_{j=1}^{m} v_j(t)\cos\eta_j=\sum_{j=1}^{m} s_j'(t)\cos\eta_j=z'(t)\end{aligned} \tag{14}$$

which is the general law of decomposition of any number of speeds in three orthogonal speeds.[129]

In exactly the same way, we can deduce the law of composition of forces. By eliminating ϑ in the three equations (8), we have:

$$\mathrm{Y}(\vartheta)=\mathrm{X}(\vartheta)\frac{y''(t)}{x''(t)};\ \mathrm{Z}(\vartheta)=\mathrm{X}(\vartheta)\frac{y''(t)}{x''(t)} \tag{15}$$

which also express a three-dimensional straight line passing through point T. So, all previous *formulæ* can be repeated simply by exchanging $v(t)$ with $u(t)$ (which express the measure of the force acting upon the body in instant t); ϑ with $(1/2)\vartheta^2$; $x'(t)$, $y'(t)$, $z'(t)$ respectively with $x''(t)$, $y''(t)$, $z''(t)$; and ε, ζ, η respectively with α, β, γ (which denote the angles of the straight line expressed by the equation (15) with the axes x, y, z).

Thus, the instantaneous acceleration of the moving body upon which m forces act is:

$$u(t)=\sqrt{\left[\sum_{j=1}^{m} x_j''(t)\right]^2+\left[\sum_{j=1}^{m} y_j''(t)\right]^2+\left[\sum_{j=1}^{m} z_j''(t)\right]^2} \tag{16}$$

(where $x_j''(t)$, $y_j''(t)$, $z_j''(t)$ express the instantaneous acceleration of component motions $x_j=x_j(t)$, $y_j=y_j(t)$, $z_j=z_j(t)$); and the law of composition of forces is expressed by the identities:

$$\begin{aligned} u(t)\cos\alpha&=\sum_{j=1}^{m} u_j(t)\cos\alpha_j=x''(t)\left[=\sum_{j=1}^{m} x_j''(t)\right]\\ u(t)\cos\beta&=\sum_{j=1}^{m} u_j(t)\cos\beta_j=y''(t)\left[=\sum_{j=1}^{m} y_j''(t)\right]\\ u(t)\cos\gamma&=\sum_{j=1}^{m} u_j(t)\cos\gamma_j=z''(t)\left[=\sum_{j=1}^{m} z_j''(t)\right]\end{aligned} \tag{17}$$

where $u(t)$, $x''(t)$, $y''(t)$ and $z''(t)$ are the measures of the effective total forces.[130]

Let $\Delta_{v,u}$ now be the angle formed in T by the directions of force (measured by) $u(t)$ and speed $v(t)$. By very simple geometric considerations, we have:

$$\cos \Delta_{v,u} = \cos \varepsilon \cdot \cos \alpha + \cos \zeta \cdot \cos \beta + \cos \eta \cdot \cos \gamma \tag{18}$$

and so, multiplying the identies (17) respectively by $\cos \varepsilon$, $\cos \zeta$ and $\cos \eta$ and adding, we deduce (according to (12)):

$$\begin{aligned} x''(t) \cos \varepsilon + y''(t) \cos \zeta + z''(t) \cos \eta &= \\ &= \frac{x''(t)x'(t) + y''(t)y'(t) + z''(t)z'(t)}{v(t)} = v'(t) = s'(t) \\ &= u(t) \cos \Delta_{v,u} = \sum_{j=1}^{m} u_j(t) \cos \Delta_{v,u_j} \end{aligned} \tag{19}$$

η. Here, [131] the forces are measured by the effects they produce on bodies. Generally, this effect depends on the masses of the body, consequently, relative to a body with mass M, the effective forces are given by M$u(t)$, M$x''(t)$, M$y''(t)$, M$z''(t)$.

In previous works, Lagrange denotes by P, Q, &c. the effects of forces on bodies (accelerative forces) so, to give the general principle of virtual velocities, he introduces masses M_i ($i = 1, 2, \ldots, n$) as common factors. Contrarily, he now notes by P, Q, &C. "absolute forces", thus, introducing the masses, we have to replace the symbols $u_1(t)$, $u_2(t)$, &c. with P/M, Q/M, &c..

To clarify the correspondence between the present and previous *formulæ*, I have not adopted Lagrange's notation.[132] Therefore, to refer (17) and (19) to effective forces acting upon the considered body (and not simply to its measures) we have only to multiply the two members of the equations by mass M. If we do not consider anything but one body, we can always take M = 1 and repeat the same equations (17) and (19) relative to effective forces too.[133] However, to make clear the relation between these equation and that of virtual velocity for a discrete system, I shall introduce the factor M, so that (19) becomes:

$$\mathrm{M}[x''(t)x'(t) + y''(t)y'(t) + z''(t)z'(t)] - \mathrm{M}\left[\sum_{j=1}^{m} u_j(t)v(t) \cos \Delta_{v,u_j}\right] = 0 \tag{20}$$

Therefore, even if Lagrange does not mention it, (19) contains the general principle of virtual velocities for discrete systems. Really, in Lagrange's previous mechanical works, the virtual velocity of a body placed at (x, y, z) was measured, by a variation δs composed by three independent variations δx, δy, δz of its positional variables. If x, y and z are taken as arbitrary and independent functions of time, we can consider these variations as the increments of such functions produced during time $\delta t = \vartheta$. Thus we have: $\delta x = x'(t)\delta t$, $\delta y = y'(t)\delta t$, $\delta z = z'(t)\delta t$ and $\delta s = s'(t)\delta t$. Projections of the virtual velocity of the body on axes x, y, and z can be therefore represented simply by the first derivatives of $x = x(\tau)$, $y = y(\tau)$ and $z = z(\tau)$ at point

$T(x(t), y(t), z(t))$. Now, according to (I. 33), posing $P = u_1(t)$, we have: $\delta p = \delta x \cos \alpha_1 + \delta y \cos \beta_1 + \delta z \cos \gamma_1 = [x'(t) \cos \alpha_1 + y'(t) \cos \beta_1 + z'(t) \cos \gamma_1]\delta t = [v(t) \cos \Delta_{v,u_1}]\delta t = u_1'(t)\delta t$. Then, the virtual velocity of the body placed at T "evaluated along the direction of force $Mu_j(t)$" can be represented by the first derivative relative to time of $u_j(t)$ at this point—i.e. $u_j'(t) = v(t) \cos \Delta_{v,u_j}$—taken negatively (the force tending to diminish the distance between the point and the origin of force).

Thus, by considering any number n of bodies upon which different forces act, and by simple introduction in (20) of a new index, we have a version of the principle of virtual velocities for discrete systems in terms of derivative functions:

$$(21) \qquad \sum_{i=1}^{n} M_i \left[x_i''(t)x_i'(t) + y_i''(t)y_i'(t) + z_i''(t)z_i'(t) + \sum_{j=1}^{m} u_{i,j}(t)[-u_{i,j}'(t)] \right] = 0$$

where $M_i u_{i,1}(t)$, $M_i u_{i,2}(t)$, &c. are different forces acting upon the mass-point of mass M_i and position $T_i(x_i(t), y_i(t), z_i(t))$ and $(-u_{i,1}'(t))$, $(-u_{i,2}'(t))$, &c. are virtual velocities of the bodies placed at T_i "evaluated along the directions of forces" $M_i u_{i,1}(t)$, $M_i u_{i,2}(t)$, &c. $(i = 1, 2, \ldots, n)$.

Therefore, introducing a temporal parameter, we may express the dynamic version of the principle of virtual velocities for discrete systems according to Lagrange's interpretation in *Méchanique analytique*, without using any specific variational symbol. Independent variables, of which the indeterminate coefficients are to equate to zero, are the first derivatives of arbitrary independent motions expressed by functions of a common temporal parameter. So, the general method of indeterminate coefficients can be also applied by starting from this new version of the principle.

θ. The general *formulæ* of the previous paragraph ζ can, however, be applied to solve usual dynamic problems of discrete systems only if we can transform the expressions of speeds and forces in terms of derivative functions by eliminating the external variable t. In fact, the position of a body is usually given by a functional relation between the positional variables x, y, z and, if virtual velocities must be independent of this functional link (to apply the general method of indeterminate coefficients), this is not the case for forces.

Thus, an essential tool to make the new derivative representation of speed and force work is a general method of elimination of time.

We have, here, a particular case of the general problem to provide the analytical relations between the derivatives of any function $f(x)$ relative to x and derivatives of the same function relative to any function of x, $g(x)$. This problem is approached in all its generality by Lagrange only in lecture VII of his *Leçons sur le calcul des fonctions*, which constitute a commentary on *Theorie*[134] and which were published four years later. If function is an analytical form, as in Lagrange's conception, to change the reference variable is strictly to change the function. Therefore, to consider different derivatives of the *same* function relative to different functions taken as variables, we have to take functions as objects which remain the same even by changing their analytical form, i.e. not simply as formal objects. Such objects are quantities or non analytical (i.e.: geometric or mechanical) entities. The *leçon* devoted to this problem is among the most problematic for Lagrange's general point of view and it was probably originated by a consideration of particular (not foundational) problem of elimination of variables.

A similar problem—showing that traditional mathematics present some aspects which are not strictly reconstructable in Lagrange's radical point of view, and using Lagrange's concept of function—turns up on two different occasions in the *Théorie*. The first occasion is the passage from the theory of one-variable primitive functions to the theory of two-variables primitive functions.[135] The second occasion is just the method of elimination of time from equations of motion. Here the external entities are, evidently, the positional coordinates of the point T which remain the same considered either as independent functions of time or linked among them by a functional relation. So, their different derivatives relative to t or to x are different analytical objects, which maintain some standard analytical relations with not analytical objects. The problem is just to determine these analytical relations, taking x, y, z as not analytical objects, i.e.: *as quantities* and *not as forms*.

It is perhaps because of these general conceptual difficulties that Lagrange does not recall the general *formulæ* proven in his first treatment of the problem of elimination of variables, but proves them again with a new procedure.[136]

If x and y are considered—as before—arbitrary functions of time, marking explicitly the reference variables of derivations, we have:[137]

$$\begin{aligned}(22)\qquad x(t+\vartheta)&=x(t)+x_t'(t)\vartheta+(\vartheta^2/2!)x_t''(t)+(\vartheta^3/3!)x_t'''(t)+\&c\,.\\ y(t+\vartheta)&=y(t)+y_t'(t)\vartheta+(\vartheta^2/2!)y_t''(t)+(\vartheta^3/3!)y_t'''(t)+\&c\,.\end{aligned}$$

Moreover, if we consider y as a function of x, noting by ξ an arbitrary increment of x, we have:

$$(23)\qquad y(x+\xi)=y(x)+y_x'(x)\xi+(\xi^2/2!)y_x''(x)+(\xi^3/3!)y_x'''(x)+\&c\,.$$

ξ being arbitrary, we can put: $\xi=\xi(t)=x_t'(t)\vartheta+(\vartheta^2/2!)x_t''(t)+(\vartheta^3/3!)x_t'''(t)+\&c.$, so that $t+\vartheta$ and $x+\xi$ mark corresponding increments and, then:

$$(24)\qquad y(t+\vartheta)-y(t)=y(x+\xi)-y(x)\,;$$

that is:

$$y_t'(t)\vartheta+(\vartheta^2/2!)y_t''(t)+\&c.=y_x'(x)\xi+(\xi^2/2!)y_x''(x)+\&c\,.$$

Replacing ξ with its value and equating the coefficients of different powers of indeterminate ϑ to zero, we have:

$$\begin{aligned}(25)\qquad & y_t'(t)=y_x'(x)x_t'(t)\\ & y_t''(t)=y_x'(x)x_t''(t)+y_x''(x)[x_t'(t)]^2\\ & y_t'''(t)=y_x'(x)x_t'''(t)+3y_x''(x)x_t'(t)x_t''(t)+y_x'''(x)[x_t'(t)]^3\\ & \&c\,.\end{aligned}$$

Here $x_i'(t)$ is the initial velocity (virtual velocity) of the body along direction of axis x and $x=x(t)$ is taken as independent variable in $y=y(x)$. The same deduction can obviously be repeated relative to z.

ι. In Lagrange's new approach, the general dynamic equation of a system of bodies is deduced starting by initially considering an isolated body. Thus, the generic equations of motion of a body—which, according to the general method of *Méchanique analitique*, results from equating the coefficients of virtual velocities of this body to zero, after having introduced the equations of condition of the system —now becomes the first step of the deduction. Consequently, the first example Lagrange gives is the solution to a one-body problem.

The selected problem is to find the punctual resistance of a *medium* in order that a body thrown into it describes a given curve. It is Problem III (proposition X) of Newton's second book of *Principia*. Newton's solution in the first edition of *Principia*[138] is, however, wrong and the mistake was pointed out by Johann I and Niklaus Bernoulli in 1711,[139] although the former does not show the error in Newton's reasoning and the latter claims a correct passage to be wrong. A new and correct solution was given by Newton in the second edition of the *Principia*,[140] where he changed the demonstrative method, and then did not point out the effective error in his previous reasoning.

Having solved the problem by two alternative procedures, Lagrange thoroughly analyses Newton's first solution in order to find its mistake and to reconstruct his method of series.[141] Both Lagrange's solutions are simply applications of general identities (25) to eliminate t from the specific equations of the problem coming from (17):

$$x''(t) = -\mathrm{R}\cos\varepsilon\,;\; y''(t) = -\mathrm{R}\cos\zeta - g\,;\; z''(t) = -\mathrm{R}\cos\eta \tag{26}$$

where ε, ζ and η are the angles of the tangent to the trajectory of the body with the axes x, y, z; R is the ratio between the resistance (considered a negative force acting upon the body along the direction of this tangent) and the mass of the body; g is the force of gravity (which is independent of the mass of the body); and the coordinates system is oriented so that the gravity acts along the direction of axis y.

The example shows that in Lagrange's new method, if (17) is given, the solution of a one-body dynamic problem is essentially reduced to a formal procedure of transforming derivative equations relative to one variable into corresponding derivative equations relative to another variable.

κ. Let us return[142] to equations (17). These are the general equations expressing the conditions of composition of forces. To apply them, we have to consider explicitly all the forces acting upon the body. Let us consider first a free moving body upon which m central forces $u_j(t)$ $(j=1, 2, \ldots, m)$ act. If the position both of this body and of the origins of forces are expressed in orthogonal coordinates x, y, z, equations (17) can be simply applied after having eliminated trigonometric expressions. If we consider the points (a_j, b_j, c_j) $(j=1, 2, \ldots, m)$ as the origins of forces $u_j(t)$ and the body placed in point T and we denot by p_j the distances between T and the points (a_j, b_j, c_j), we have (from geometrical considerations):

$$\cos\alpha_j=\frac{x(t)-a_j}{p_j};\quad \cos\beta_j=\frac{y(t)-b_j}{p_j};\quad \cos\gamma_j=\frac{z(t)-c_j}{p_j} \tag{27}$$

On the other hand since

$$\sqrt{[(x)-a_j]^2+[y(t)-b_j]^2+[z(t)-c_j]^2}-p_j=0 \tag{28}$$

$(j=1, 2, \ldots, m)$ are the equations of the spheric-surfaces with centres (a_j, b_j, c_j) and *radii* p_j, if we denote by ${}_j\mathrm{S}(x, y, z)=0$ these equations, we can replace in (17) $\cos\alpha_j$, $\cos\beta_j$ and $\cos\gamma_j$ respectively with ${}_j\mathrm{S}'_x(x, y, z)$, ${}_j\mathrm{S}'_y(x, y, z)$ and ${}_j\mathrm{S}'_z(x, y, z)$.[143] Thus, the seconds members of equations (17) will take the form $\sum_{j=1}^{m} u_j(t)\,{}_j\mathrm{S}'_\varpi(x, y, z)$ $(\varpi=x, y, z)$.

If the moving body is not free, i.e. its motion is submitted to some constraints relative to its possible trajectory, in order to apply (17) we can try to transform the analytical expressions of this constrains into analytical expressions of appropriate forces acting upon it. To do this, Lagrange uses a standard procedure—the so called "method of multiplicators"—just set forth in the *Méchanique analitique*,[144] where it was considered as an alternative (and sometimes simpler) procedure to "immediate elimination" from the equation of *equilibrium* of an arbitrary n-bodies system, of dependent variables relative to peculiar equations of condition of the studied particular system.

Suppose the body is forced to move on a given surface $\mathrm{W}(x, y, z)=0$. We can always consider such a condition as the presupposition of an "action" of this surface on the body, i.e.: as a presupposition of a central force acting perpendicularly to the tangent sphere $\mathrm{S}(x, y, z)=0$. Now, as a geometric result,[145] we know that the partial derivatives $\mathrm{W}'_x(x, y, z)$, $\mathrm{W}'_y(x, y, z)$ and $\mathrm{W}'_z(x, y, z)$ are respectively proportional to the corresponding partial derivatives of $\mathrm{S}(x, y, z)$, thus, according to previous reasoning relative to central forces, to introduce the condition that the body has to move on this surface, is analytically equivalent to introducing in the second members of three equations (17) respectively the terms $\Pi\mathrm{W}'_x(x, y, z)$, $\Pi\mathrm{W}'_y(x, y, z)$, $\Pi\mathrm{W}'_z(x, y, z)$ proportional to $\mathrm{W}'_x(x, y, z)$, $\mathrm{W}'_y(x, y, z)$, $\mathrm{W}'_z(x, y, z)$.

In general, we can always consider any equation of condition like $\mathrm{F}(x, y, z)=0$ as an analytical representation of a surface on which the body is forced to move. Thus, to introduce the corresponding condition is equivalent to introduce a central force directed along the *radius* of a sphere tangent to this surface at point T, i.e.: to introduce in the second members of three equations (17) respectively the terms $\Pi\mathrm{F}'_x(x, y, z)$, $\Pi\mathrm{F}'_y(x, y, z)$ and $\Pi\mathrm{F}'_z(x, y, z)$, where Π is an indeterminate coefficient to eliminate in order to solve the problem.

λ. Even if I have shown the analytical correspondence between Lagrange's *formulæ* relative to an isolated body with the general equation of virtual velocities, Lagrange has, till now, considered only isolated bodies. His programme is, really, to deduce the general equation of a system, starting from (17), without presupposing the general equation of principle of virtual velocities. Thus, instead of starting from this equation and reducing its terms by the introduction of functional links given by the equations of condition, in order to arrive at the specific equations of the studied system, he wants to determine *directly* the complete equations of motion of each body in such a system, starting from the equations of condition expressing its particular mechanical characteristics.

A system of moving bodies can be characterized by specification of internal and external central attractive forces acting upon the bodies and by the internal constraints for their absolute or relative motion. As we have just seen, a central force can be analytically represented by the product of a coefficient—representing the intensity of the force (relative to the mass of the body, that we can not consider any more as unitary)—for a partial derivative of a function like (28)—expressing its direction. Thus, if a central external force $M_\nu u_\upsilon(t)$ act upon the υ-th body of the system with position (x_ν, y_ν, z_ν) and mass $M_\nu (0 < \upsilon < m;\ 0 < \nu < n)$, we can deduce the relative terms in the second member of equations (17) for this body by an equations of condition of the form:

$$\sqrt{(x_\nu - a_\upsilon)^2 + (y_\nu - b_\upsilon)^2 + (z_\nu - c_\upsilon)^2} - p_{\nu,\upsilon} = 0 \tag{29}$$

where $p_{\nu,\upsilon}$ is the variable distance of the body from the point (a_ν, b_ν, c_ν) which is the origin of force. These terms will be nothing but the products of the partial derivatives of the first member of such an equation (relative to variables x_ν, y_ν, z_ν explicitly occurring in it) and a coefficient expressing the intensity of force (relative to the mass of the body).

Equally, for internal central forces, simply replacing in (29) the fixed coordinates a_ν, b_ν, c_ν by the position coordinates of the other body of the system which is the origin of force.

On the other hand, any internal relative or absolute restraint in the system can be expressed as a relation between the position coordinates of some body (that is: one body, if the restraint is relative to absolute trajectory of the body, or a number of bodies if it refers to relative positions of a number of bodies). Let (x_ν, y_ν, z_ν), (x_μ, y_μ, z_μ), &c. be the positions of a number of bodies in the system. The typical form of the equations of condition is:

$$f(x_\nu, y_\nu, z_\nu, x_\mu, y_\mu, z_\mu, \&c.) = 0 \tag{30}$$

Now, because of

> la direction des forces [...] doit être la même dans un instant quelconque, soit que les corps se meuvent ou non, puisqu'elle dépend uniquement de la disposition mutuelle des corps dans cet instant[146]

the equation (30) will give independent terms in second members of equations (17) for each body of which the position-coordinates occurring in it and these terms will be given by the product of a common coefficient for the partial derivatives of $f(x_\nu, y_\nu, z_\nu, x_\mu, y_\mu, z_\mu, \&c.)$ relative only to position-coordinates of such a body. Thus, if (30) is effectively an equation of condition of the system, we can deduce from it the following terms of second members of equations (17) for the bodies which positions are (x_ν, y_ν, z_ν), (x_μ, y_μ, z_μ), &c.:

$$\Pi[f'_{x_\nu}(x_\nu, \&c.)]\,, \quad \Pi[f'_{x_\nu}(x_\nu, \&c.)]\,, \quad \Pi[f'_{z_\nu}(x_\nu, \&c.)]$$
$$\text{[for the body placed at } (x_\nu, y_\nu, z_\nu)]$$

$$\Pi[f'_{x_\mu}(x_\nu, \&c.)]\,, \quad \Pi[f'_{x_\mu}(x_\nu, \&c.)]\,, \quad \Pi[f'_{x_\mu}(x_\nu, \&c.)] \tag{31}$$
$$\text{[for the body placed at } (x_\mu, y_\mu, z_\mu)]$$

&c.

Thus, we can consider a mechanical discrete system completely determined by a set of equations of condition of the three following forms:

(32) i) $f(x_\nu, y_\nu, z_\nu, p_\nu) = \sqrt{(x_\nu - a_\nu)^2 + (y_\nu - b_\nu)^2 + (z_\nu - c_\nu)^2} - p_\nu = 0$

ii) $f(x_\nu, y_\nu, z_\nu, x_\mu, y_\mu, z_\mu, p_{\nu,\mu}) = \sqrt{(x_\nu - x_\mu)^2 + (y_\nu - y_\mu)^2 + (z_\nu - z_\mu)^2} - p_{\nu,\mu} = 0$

iii) $f(x_\nu, y_\nu, z_\nu, x_\mu, y_\mu, z_\mu, \&c.) = 0$

Equations of type (32. i) express external attractive forces; equations of type (32. ii) express internal attractive forces, and equations of type (32. iii) express internal constraints of the system.

If the system is characterized in this way, we can determine the effective equations (17) for each body directly from equations (32). In fact, to have the second member of such an equation, it is sufficient to add among them the partial derivatives relative to the position variables of the considered body of each of these equations, where these coordinates occur, after having multiplied them by a coefficient (expressing the intensity of force relative to the mass of the body), which is the same for all derivatives taken from the same equation.

In this way, we have exactly the same results we would have had, starting from the equation of virtual velocities of the system and equating the coefficients of variations separately to zero, as was been proposed by Lagrange in *Méchanique analitique*.

As an example let us consider a completely free system of two bodies respectively of mass M_1 and M_2 and position (x_1, y_1, z_1), (x_2, y_2, z_2), each of them is attracted by one external central force and by an internal central force the origin of which is the other body. Such a system will be characterized by two equations of type (32. i) and one equation of type (32. ii). The corresponding terms in seconds members of equations (17) for the two bodies will be:

(33)

$$\text{i)}\begin{cases}\Phi_{1,1}\dfrac{x_1-a_1}{p_1},\quad \Phi_{1,1}\dfrac{y_1-b_1}{p_1},\quad \Phi_{1,1}\dfrac{z_1-c_1}{p_1} & \text{[for the body placed at } (x_1, y_1, z_1)\text{]}\\ \text{and} & \\ \Phi_{1,2}\dfrac{x_2-a_2}{p_2},\quad \Phi_{1,2}\dfrac{y_2-b_2}{p_2},\quad \Phi_{1,2}\dfrac{z_2-c_2}{p_2} & \text{[for the body placed at } (x_2, y_2, z_2)\text{]}\end{cases}$$

$$\text{ii)}\begin{cases}\Psi_{1,2}\dfrac{x_1-x_2}{p_{1,2}},\quad \Psi_{1,2}\dfrac{y_1-y_2}{p_{1,2}},\quad \Psi_{1,2}\dfrac{z_1-z_2}{p_{1,2}} & \text{[for the body placed at } (x_1, y_1, z_1)\text{]}\\ \text{and} & \\ \Psi_{2,1}\dfrac{x_2-x_1}{p_{2,1}},\quad \Psi_{2,1}\dfrac{y_2-y_1}{p_{2,1}},\quad \Psi_{2,1}\dfrac{z_2-z_1}{p_{2,1}} & \text{[for the body placed at } (x_2, y_2, z_2)\text{]}\end{cases}$$

where $\Phi_{1,1}$, $\Phi_{1,2}$, $\Psi_{1,2}$ and $\Psi_{2,1}$ express the intensities (relative to the masses of considered bodies) of the external and internal forces acting upon the two bodies; i.e.: $\Phi_{1,1}$, the intensity of the external force acting upon the first body, $\Phi_{1,2}$ the intensity of the external force acting upon the second body, $\Psi_{1,2}$ the intensity of the internal force acting upon the first body (and originated by the second one), $\Psi_{2,1}$ the intensity of the internal force acting upon the second body (and originated by the first one).

Thus if we add among each other the six equations (17) relative to our system after having introduced the values of masses, we shall have (because for $\Psi_{1,2}=-\Psi_{2,1}$ and $p_{1,2}=p_{2,1}$):

(34) $$\Phi_{1,1}(p_1)'+\Phi_{1,2}(p_2)'+\Psi_{1,2}(p_{1,2})' = \mathrm{M}_1[x_1''(t)+y_1''(t)+z_1''(t)] + \mathrm{M}_2[x_2''(t)+y_2''(t)+z_2''(t)]$$

where the derivatives $(p_1)'$, $(p_2)'$, $(p_{1,2})'$ are derivatives relative to position coordinates.[147]

If we multiply the forces along the directions of axes for the speeds of the points, we have:

(35) $$\Phi_{1,1}(p_1)_t'+\Phi_{1,2}(p_2)_t'+\Psi_{1,2}(p_{1,2})_t'= =\mathrm{M}_1[x_1''(t)x_1'(t)+y_1''(t)y_1'(t)+z_1(t)z_1'(t)]+\mathrm{M}_2[x_2''(t)x_2'(t)+y_2''(t)y_2'(t)+z_2''(t)z_2'(t)]$$

where $(p_1)_t'$, $(p_2)_t'$ and $(p_{1,2})_t'$ are total derivatives of the respective distances relative to time.

It is clear that, for a system as the one we have considered, (35) corresponds to the general equation of virtual velocities (I. 18), simply by putting: $\mathrm{M}_1\mathrm{P}_1$ for $\Phi_{1,1}$, $\mathrm{M}_2\mathrm{P}_2$ for $\Phi_{1,2}$, $2\mathrm{M}_1\mathrm{Q}_1=-2\mathrm{M}_2\mathrm{Q}_2$ for $\Psi_{1,2}=-\Psi_{2,1}$ and $q_1=q_2$ for $p_{1,2}$.

μ. Lagrange's procedure is not, however, completely independent of the general principle of virtual velocities. To prove that the coefficient (expressing the relative intensity of force) which has to be to multiplied by the partial derivatives of the equations of condition is always the same for any derivative coming from the same equation, Lagrange recalls, in fact, the statical version of the principle.

If this principle is applied, the difference between the general method of the *Méchanique analitique* and the previous one is simply a difference between two alternative applications of the same general principle and results; and it is purely expository.

However, a similar remark applies only to the first edition of the *Théorie*. Really, in the second edition Lagrange *proves* the statical version of the principle.[148] The proof he proposes is an analytical translation of the one given in the second edition of the *Mécanique analytique* but it follows by a relatively simply application of the new general method and does not require an *a priori* assumption of another principle.[149]

Let us consider a system of two bodies placed in (x_1, y_1, z_1), (x_2, y_2, z_2) and linked by a string passing through a fixed pulley, placed in the point (a_1, b_1, c_1). This restraint is expressed by the following equation:

$$(36)\quad \sqrt{(x_1-a_1)^2+(y_1-b_1)^2+(z_1-c_1)^2}+\sqrt{(x_2-a_1)^2+(y_2-b_1)^2+(z_2-c_1)^2}-\mathrm{L}=0$$

where constant L is the length of the string. If we represent (36) with (30), the second members of six equations (17) relative to the two bodies of the system have to contain the terms (31) ($v=1$, $\mu=2$). Here, the coefficient Π will be constant and expresses the string tension; thus it is evidently the same for the six terms corresponding to the equation (36).

If the string passes through two fixed pulleys respectively placed in (a_1, b_1, c_1) and (a_2, b_2, c_2), we have the same equation (36) with simple replacements of a_1, b_1, c_1 with a_2, b_2, c_2 in the second root, L now being the length of the string minus the distance between the two pulleys. Thus, the same terms (31) ($v=1$, $\mu=2$) (where f represent the first members of the new equation) have again to occur in the second members of the equations (17).

Now, let the string go from the first body to the first pulley and back again h times and then pass to the second pulley and on to the second body and back, k times. The equation of condition will be:

$$(37)\quad \begin{aligned} &h\sqrt{(x_1-a_1)^2+(y_1-b_1)^2+(z_1-c_1)^2}\\ &\quad+k\sqrt{(x_2-a_2)^2+(y_2-b_2)^2+(z_2-c_2)^2}-\mathrm{L}=0\end{aligned}$$

where L is, again, the length of the string minus the distance between the pulleys.

If we represent this equation again with (30) the same previous terms (where f is the first member of (37)) have to occur in the second member of the equations (17), the string tension being the same for every string extension.

Now, (if the masses of the two point are unit and) P and Q are two forces attracting the two bodies toward the pulleys, we shall have $\mathrm{P}=h\mathrm{U}$ and $\mathrm{Q}=k\mathrm{U}$, U being the common measure of P and Q. If P and Q are not commensurable, they can always be represented by two products hU and kU, taking infinitely small U and infinitely great h and k.

For every equation of condition $\Gamma=0$ between six variables of position (i.e: for every equation of condition relative to two bodies), we can change the arbitrary constant in (37), so that, the punctural values of Γ and Γ' coincide respectively with the punctual values of the first member of (37) and of its first derivative. Thus, the two equations $\Gamma=0$ and (37) become tangents to each other and every condition expressed by an equation like (30) (without &c.) can be considered analogous to the condition of a string passing through two pulleys. Thus, in a system of two bodies, every condition like (30) (without &c.) produces the terms (31).

The same reasoning being possible for a system of three, four, &c. bodies, Lagrange concludes that

> les forces qui peuvent résulter de l'action mutuelle des corps d'un système donné, se déduisent directement des équations de condition qui doivent avoir lieu entre les coordonnées des différens corps du système, en prenant les functions primes des fontions qui sont nulles en vertu de ces équations.[150]

Starting from this result, we can prove the general principle of virtual velocities for the equilibrium of a discrete system.

If M_iX_i, M_iY_i, M_iZ_i $(i=1, 2, \ldots, n)$ are $3n$ forces acting upon n bodies placed at points (x_i, y_i, z_i) along the directions of their orthogonal coordinates and $\Theta_i=0$, $\Xi_i=0$, &c. are the equations of condition between these bodies, we have, according to the general method:

$$\begin{aligned} &M_iX_i=\Phi_i(\Theta_i)'_{x_i}+\Psi_i(\Xi_i)'_{x_i}+\&c. \\ (38)\qquad &M_iY_i=\Phi_i(\Theta_i)'_{y_i}+\Psi_i(\Xi_i)'_{y_i}+\&c. \quad [i=1,2,\ldots,n] \\ &M_iZ_i=\Phi_i(\Theta_i)'_{z_i}+\Psi_i(\Xi_i)'_{z_i}+\&c. \end{aligned}$$

and thus

$$(39)\qquad \sum_{i=1}^{n} M_i[X_ix'_i+Y_iy'_i+Z_iz'_i]=\sum_{i=1}^{n} \Phi_i(\Theta_i)'+\Psi_i(\Xi_i)'+\&c.$$

but, because of the equations of condition, the second member of this equation is null; therefore the first member is equal to zero, and this expresses precisely the static version of the *principle of virtual velocities* for discrete systems.

This first analytical proof of the general principle of virtual velocities for equilibrium of discrete systems by Lagrange, even if it uses new mathematical tools and methods yet, does not differ substantially from mathematical reasonings considered in the previous part I of this paper. In particular, it is not very different from the proof I have outlined in the paragraph I.σ, according to Lagrange's principles and definitions.

Effectively, if we consider it deeply enough, the general theory is not really changed from the first formulation of 1764.

v. In the *Théorie*, as in his previous works, Lagrange wants to prove the deductive power of his method by an analytical deduction of the main principles of mechanics of discrete systems. Therefore, the last pages of Lagrange's treatise are devoted to the deduction and discussion of: the *principle of conservation of motion of centre of gravity*, the *law of areas* and the *principle of conservation of vis viva* for discrete systems.

The demonstrative procedures are not very different from the previous ones. I shall limit myself to outlining them.

To prove the first principle, let us assume that the system is completely free to move along the direction of x axis. It is clear that this condition means that the equations of condition expressing the internal constraints to the system are completely independent of the origin of this axis, i.e.: the functions making up the first members of these equations are such that

> si on augmente à la fois les abscisses [...] des différens corps, d'une même quantité quelconque ξ, cette quantité disparaisse d'elle-même des fonctions.[151]

Thus, taking (30) as a typical equation of condition of such a system, we have to assume that all the factors of successive powers of ξ in the development of $f(x_\nu+\xi, y_\nu, z_\nu, x_\mu+\xi, y_\mu, z_\mu, \&c.)$ are separately null. Considering only the first of these factor we shall have[152]:

$$f'_{x_\nu}(x_\nu, \&c.)+f'_{x_\mu}(x_\nu, \&c.)+\&c.=0 \tag{40}$$

from which it is clear that the component of the sum $\sum_{i=1}^{n} M_i[x''_i(t)]$, due to the internal constraints to the system, is null.

The same reasoning can be repeated for the component due to the internal central forces, arriving at the same conclusion. In fact, it is easy to see that (32. ii) does not change its value substituting in it x_ν and x_μ with $x_\nu+\xi$ and $x_\mu+\xi$.

Therefore, if the system is free to move along the direction of x axis, we have:

$$\sum_{i=1}^{n} M_i(x''_i(t))=\sum_{i=1}^{n} M_i X_i^e \tag{41}$$

where $M_i X_i^e$ $(i=1, 2, \ldots, n)$ denote the total external force acting along the direction of x axis upon the bodies placed in T_i with mass M_i. Hence, putting:

$$x_G(t)=\frac{\sum_{i=1}^{n} M_i x_i(t)}{\sum_{i=1}^{n} M_i} \tag{42}$$

and repeating the same reasoning for the y and z axes, we conclude that if the system is free to move along the direction of the three axes, then:

$$\sum_{i=1}^{n} M_i x''_G(t) = \sum_{i=1}^{n} M_i X_i^e$$

$$(43) \qquad \sum_{i=1}^{n} M_i y''_G(t) = \sum_{i=1}^{n} M_i Y_i^e$$

$$\sum_{i=1}^{n} M_i z''_G(t) = \sum_{i=1}^{n} M_i Z_i^e$$

Therefore, in such a system point $T_G(x_G(t), y_G(t), z_G(t))$ moves independently from the internal forces[153] and as it has mass equal to $\sum_{i=1}^{n} M_i$ and all forces of the system acted on it along their proper directions. Thus, if we call T_G the centre of gravity of the system, we have both a definition of this and a deduction of the principle of conservation of its motion.

To prove the law of areas, Lagrange starts by assuming the system is completely free to turn around the z axis, perpendicular to plane x, y and progresses exactly as in the previous case.

In fact, under this condition we can replace in all the equations of condition expressing the internal constraints on the system, x_i and y_i with $x_i \cos\chi - y_i \sin\chi$ and $y_i \cos\chi + x_i \sin\chi$ $(i = 1, 2, \ldots, n)$ χ being an indeterminate and arbitrary angle. If we pose

$$(44) \qquad \xi_i = x_i(\cos\chi - 1) - y_i \sin\chi\,; \; \omega_i = y_i(\cos\chi - 1) + x_i \sin\chi\,.$$

such replacement corresponds to the replacement of x_i and y_i respectively with $x_i + \xi_i$ and $y_i + \omega_i$, where ξ_i and ω_i can be considered indeterminate and arbitrary increments. Thus taking (30) as a generic equation of the system expressing its internal constraints, we have:

$$(45) \qquad \begin{aligned} f(x_\nu + \xi_\nu, y_\nu + \omega_\nu, z_\nu, x_\mu + \xi_\mu, y_\mu + \omega_\mu, z_\mu, \&c.) &= \\ = f(x_\nu, \&c.) + \xi_\nu f'_{x_\nu}(x_\nu, \&c.) &+ \omega_\nu f'_{y_\nu}(x_\nu, \&c.) \\ + \xi_\mu f'_{x_\mu}(x_\nu, \&c.) &+ \omega_\mu f'_{y_\mu}(x_\nu, \&c.) + \&c. \\ &+ \frac{\xi_\nu^2}{2!} f''_{x_\nu}(x_\nu, \&c.) + \&c. \\ &+ \&c. = 0 \end{aligned}$$

from which, replacing ξ_h and $\omega_h (h = \nu, \mu, \&c.)$ by values (44) and then, $\cos\chi$ and $\sin\chi$ by the series $1 - \chi^2/2! + \&c.$ and $\chi - \chi^3/3! + \&c.$ and equating the coefficient of χ to zero, we draw the following identity:

$$(46) \qquad x_\nu f'_{y_\nu}(x_\nu, \&c.) - y_\nu f'_{x_\nu}(x_\nu, \&c.) + x_\mu f'_{y_\mu}(x_\nu, \&c.) - y_\mu f'_{x_\mu}(x_\nu, \&c.) = 0$$

The component of the sum $\sum_{i=1}^{n} M_i[y''_i(t)x_i(t) - x''_i(t)y_i(t)]$ due to the internal constraints to the system is, then, null and the same is true for the component due to the internal central forces.

Let us now consider some external forces acting upon the points $T_i(x_i, y_i, z_i)$ $(i=1, 2, \ldots, n)$. According to the general method, the equations expressing these forces produce a number of terms of the form $\Phi[(x_i-a)/p]$ and $\Phi[(y_i-b)/p]$ (a, b and p being appropriate constants) in the first two equations (17) for each of these bodies. So if some external forces act upon the bodies of the system, the sum $\sum_{i=1}^{n} M_i[y_i''(t)x_i(t)-x_i''(t)y_i(t)]$ cannot be generally null. Nevertheless, this is precisely the case if the external forces acting upon T_i are parallel to z axis or directed towards a point of this axis.[154]

It is evident that the same reasoning may be repeated for rotations around y and x axes. So, by integrating, it is trivial to conclude that, if the system is free to turn around the origin of the axes and if the only external attractive forces acting upon the bodies of the system originate in it, independently of any internal force, we have:

$$\tag{47} \begin{gathered} \sum_{i=1}^{n} M_i[y_i'(t)x_i(t)-x_i'(t)y_i(t)]=C_1 \\ \sum_{i=1}^{n} M_i[x_i'(t)z_i(t)-z_i'(t)x_i(t)]=C_2 \\ \sum_{i=1}^{n} M_i[z_i'(t)y_i(t)-y_i'(t)z_i(t)]=C_3 \end{gathered}$$

which, in the language of derivative functions, expresses the law of areas.[155]

The new demonstration of the principle of conservation of *vis viva* too is not really different from the previous ones.

Let us first consider a system of bodies joined by a number of material links, upon which no central internal or external force acts. It is completely determined by a set of equations such as (30) and the horizontal components of the motions of its bodies are expressed by n equations of the following form:

$$\tag{48} M_i x_i''(t)=M_i[(A_i)(f_{x_i}(x_i, \&c.)+(B_i)(g'_{x_i}(x_i, \&c.)+\&c.] \quad [i=1, 2, \ldots, n]$$

$f(x_i, \&c.)=0$, $g(x_i, \&c.)=0$, &c. being the equations of conditions relative to each body and A_i, B_i, &c. a number of appropriate coefficients. Because these coefficients are always the same for every partial derivative coming from the same equations of conditions, calculating the correspondent values of $M_i y_i''(t)$ and $M_i z_i''(t)$ we have[156]:

$$\tag{49} \begin{aligned} &\sum_{i=1}^{n} M_i[x_i''(t)x_i'(t)+y_i''(t)y_i'(t)+z_i''(t)z_i'(t)] \\ &\qquad = M_i[A_i f_t'(x_i, \&c.)+B_i f_t'(x_i, \&c.)+\&c.]=0 \end{aligned}$$

or, integrating,

$$\tag{50} \sum_{i=1}^{n} M_i[v_i(t)]^2=2C_1$$

which clearly expresses the conservation of *vis viva* for all discrete systems in which the bodies

> n'éprouvent d'autres actions que celles qui résultent de leur liaison, et, en général, de toute les conditions qui peuvent être experimées par des équations entre les différentes coordonnées du corps, sans que le temps y entre.[157]

Now, let us consider a generic mechanical system where internal attractive forces act upon the bodies. The equations of condition expressing these forces are of the form (32. ii) and, according to (33) and (35), it is not hard to draw the identity:

$$\text{(51)} \qquad \sum_{i=1}^{n} \mathrm{M}_i[x_i''(t)x_i'(t)+y_1''(t)y_i'(t)+z_i''(t)z_i''(t)] = \sum_{r=1}^{n}\left[\sum_{s=1}^{n} \mathrm{M}_r \Psi_{r,s}(p_{r,s})_t'\right]$$

where in general: $\Psi_{r,s}=0$ if $r=s$ or if no force with origin at $\mathrm{T}_r(x_r, y_r, z_r)$ acts upon the body placed in $\mathrm{T}_s(x_s, y_s, z_s)$; $\Psi_{\mu,\nu}=\Psi_{\nu,\mu}$ and $p_{\mu,\nu}=p_{\nu,\mu}$ $(1 \le \nu, \mu \le n)$.

If external attractive forces also act on the bodies of the system, the terms deduced by the equations of type (32. i) also occur in equations (17). Therefore, according once more to (33) and (35), we have:

$$\text{(52)} \qquad \sum_{i=1}^{n} \mathrm{M}_i[x_i''(t)x_i'(t)+y_i''(t)y_i'(t)+z_i''(t)z_i'(t)] =$$

$$= \sum_{r=1}^{n}\left[\sum_{s=1}^{n} \mathrm{M}_r \Psi_{r,s}(p_{r,s})_t'\right] + \sum_{i=1}^{n}\left[\sum_{j=1}^{m} \mathrm{M}_i \Phi_{i,j}(p_{i,j})_t'\right]$$

where $\mathrm{M}_i\Phi_{i,1}$, $\mathrm{M}_i\Phi_{i,2}, \ldots, \mathrm{M}_i\Phi_{i,m}$ are the external attractive forces, eventually null, acting respectively upon the bodies placed in $\mathrm{T}_i(x_i, y_i, z_i)$ $(i=1, 2, \ldots, n)$.

It is clear—even if Lagrange does not mention it—that (52) is the general equation of virtual velocities (the virtual velocities along the directions of forces being given by the derivative of the distances taken negatively).

Moreover, if we take the forces as functions of the distances between the point upon which they act and their origins, we can also consider the generic products $\Psi_{r,s}(p_{r,s})_t'$ and $\Phi_{i,j}(p_{i,j})_t'$ as derivatives of appropriate functions of distances $p_{r,s}$ and $p_{i,j}$. Thus the entire second member of (52) can be considered a sum of derivatives of appropriate functions of all the distances occurring in it. If we denote these functions by $-\mathrm{F}_i$ $(i=1, 2, \ldots, n)$ and, for simplicity, all the distance by p, q, &c., passing to the primitive we have:

$$\text{(53)} \qquad \sum_{i=1}^{n} \mathrm{M}_i[v_i(t)]^2 = 2\mathrm{C}_2 - 2\sum_{i=1}^{n} \mathrm{M}_i\mathrm{F}_i(p, q, \&\text{c.})$$

which is clearly the equation of conservation of *vis viva* for discrete systems. The correspondence between Lagrange's new deduction of this equation and those of 1764 and 1788 is absolutely evident.

ξ. With the demonstration of the principle of conservation of *vis viva* and its applications to some particular cases, Lagrange ends his treatise. His last words clearly express the spirit of his work:

> Je ne m'étends pas davantage sur les applications á la méchanique, et je ne m'arrêterai pas à résoudre des problèmes particuliers. Comme mon desséin n'est pas de donner un Traité de mécanique, je me contente d'avoir déduit de la théorie des fonctions, les principes et les équations fondamentales du mouvement, qu'on ne démontre ordinairement que par la considération des infiniment petits.[158]

In the second edition Lagrange adds:

> [...] et d'avoir donné, d'une manière nouvelle, les lois générales du mouvement de ces corps animés par des forces quelconques, et qui agissent les uns sur les autres; et je renverrai à la *Mécanique analytique* ceux qui desireraient un plus grand détail.[159]

The reference to the *Mécanique analytique* is very natural. Far from contradicting the old approach to the foundation of mechanics, Lagrange's new exposition seems to be (even if not "literal") a translation of previous principles, methods, results and proofs.

III. Concluding Remarks

My aim in this paper was to outline the evolution of Lagrange's foundation of mechanics of discrete systems from the first work of 1760–61 to the *Théorie des fonctions analytiques*. My essential conclusion is that this evolution can be regarded as the realization of one scientific (and philosophical) programme. This programme is reductionist: mechanics of discrete systems has to be reduced to a mathematical analysis applied to some basic principles.

Both in the works of 1760–61 and in 1764 and 1780, Lagrange's starting points is an analytical formulation of *one* mechanical principle: the least action principle in the former case and the principle of virtual velocities in the latter case. On the contrary, in *Méchanique analitique*, the starting point is a "conceptual" formulation of the latter principle, so that the first essential step has to be the justification of an analytical formulation; for this purpose, Lagrange has to provide an extremely classical differential interpretation of speed and force. Mechanics of discrete systems is, then, founded on one general principle and on a mathematical interpretation of central mechanical concepts.

The philosophical aim of the *Théorie* is to show that high analysis can be completely founded on ordinary analysis, that is: *calculus* can be reduced to algebra. Instead of limiting himself to a new justification of the algorithm of old *calculus* and to a general rule of replacement of differential entities with derivative ones, Lagrange wants to reconstruct all the edifice of classical high analysis on the basis of Taylor's development. Thus it is natural that he wants to outline a possible reconstruction of one of its main branches, such as mechanics of discrete systems.

The base of a similar reconstruction is a new interpretation, in terms of derivative functions, of speed and force. The historical and conceptual reference of Lagrange's reasonings is fluxional foundation of *calculus*. Speed and force are *measured* by a rectilinear space and so are *represented* by geometric entities. But, as Maclaurin[160] —and Newton himself[161]—had shown, the essential geometric entities in the theory of curves can be represented analytically by Taylor's series terms, so that geometry (and, particularly, the geometric theory of curves) can be reduced to a general theory of Taylor's series and so to algebra. If we are able to reconstruct mechanics of discrete systems as a geometry of (spaces measuring) speeds and forces, the general goal is achieved.

The only effective difference between this programme and the foundation of mechanics of discrete systems in *Méchanique analitique* is the new interpretation of geometric entities in terms of derivative functions.

All principal ideas and mathematical tools of the previous treatise and works can be used again in the new context.

However Lagrange's translations is not literal. The place of the general "systemic" principle of virtual velocities is much less central and it is easy to understand that it can be completely avoided. Instead of deducing the equations of motion of each body of a discrete system from the general equation of this system, and by an application of the classical method of indeterminate coefficients, Lagrange directly determines these equations starting from a characterization of the system by a set of equations of condition. The method he sets up in order to fulfil this deduction is not a new one. It is just present in *Méchanique analitique* as an alternative procedure to solve mechanical problems. The principle of virtual velocities can now be deduced by adding the equations of motion of all bodies of a (completely free) system.

Particularly interesting in Lagrange's translation is the replacement of indeterminate and arbitrary variations with derivatives relative to the time. The introduction of time is, on the other hand, the essential mathematical tool allowing the new interpretation of speed and force. However, the introduction of an external parameter is strictly possible only if we are able to identify a function other than as an analytical form. Here, we can point out one of the most relevant internal problems of Lagrange's theory of analytical functions.

Finally, I think it is possible to conclude with the affirmation that Lagrange's foundation of mechanics of discrete systems is a very good example of the evolution, by translation and by changing of external (philosophical) aims, of a scientific research programme. However, I am far from thinking a general theory of mathematical research programmes and their evolution is possible. The research programme is an excellent historical notion, but it loses its explanatory force if we transform it into a general philosophical notion. Perhaps it is a consequence of the fact that philosophy of mathematics is interesting only if it is either a gnoseological theory of philosophy *in* mathematics—and a general investigation of the gnoseological characters of mathematical knowledge—or an analysis of real mathematics and its conceptual frameworks.

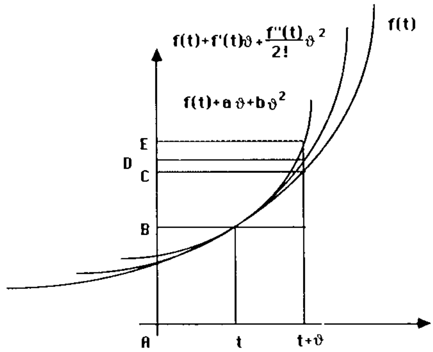
Notes

103. Cf. Lagrange (1797) [1th ed.] and (1813) [2nd ed.]. Quotations are taken from the first edition. In absence of any remarks the same passages can be found—substantially unmodified—also in the second edition. On the contrary, any important difference relative to considered problems, will be underlined.
104. Cf. Lagrange (1797), p. 5. The "circonstances particulières" to which Lagrange refers here are his lectures in 1795 and 1796 at the *Ecole Polytechnique*. [Cf., for example, de Prony (1796), p. 208 and (1835), p. 365 and *Correspondance de l'Ec. Poly.* III, 1, p. 93].
105. Cf. the complete title itself of Lagrange (1797). For the philosophical character of Lagrange's text and its mathematical purport, cf. Ovaert (1976), Fraser (1987) and Panza (1992), part III, ch. 6, sect. *a*.
106. Cf. de Prony (1835), p. 365.
107. Cf. Carnot (1813), p. 197.
108. Cf. Lagrange (1811–15), vol. I. p. VI.
109. Cf. Lagrange (1788), p. VI and (1811–15), vol. I. p. V.
110. This presupposition rests on Lagrange's concept of function. An analysis of Lagrange's presupposition and their relations with his perception of mathematical concepts is not possible here. I have treated this matter in Panza (1992), part III, ch. 6.
111. The relations between these theorems (and, even, their precise form) is relatively unclear in Lagrange's text. In the literature cf. Grabiner (1981) and Ovaert (1976). I have advanced a personal interpretation in Panza (1992), part III, ch. 6, sect. *d*. For Lagrange's text cf. Lagrange (1797), pp. 11–12, 45–50 and 226, (1813), pp. 14–5, 59–69 and 315, (1801), pp. 78–9.
112. Cf. Lagrange (1797), p. 50.
113. *ibid.*, p. 226.
114. Cf. the note (2).
115. It is clear that for a one-body problem the consideration of the mass is analytically needless, i.e. we can always take the mass of the body as unitary. Thus the forces acting upon this body can be perfectly represented by the acceleration they produce. Therefore in the following paragraphs I shall consider the masses of the bodies only relative to n-bodies ($n>1$) problems.
116. In eighteenth-century analytical mechanics two sort of principles can be distinguished: the three Newtonian laws and the other principles such as those of least action or of virtual velocities [I have insisted on this point in my (forth.), paragraph 1]. Even if the role of the principles of the second kind in the construction of an analytical mechanics of n-bodies systems was essential, their deduction from the Newtonian laws remained a philosophical exigency. Lagrange's programme in the mechanical part of *Théorie* seems to me to reply also to this philosophical requirement.
117. Cf. Lagrange (1797), pp. 223–24. On Lagrange's appeal to experience cf. the note (118).
118. Cf. *ibid.*, p. 224. As is often the case in XVIII-th century texts, the generic and isolated appeal to experience is here nothing but a rhetorical tool to avoid any philosophical discussion concerning the foundation and legitimacy of sciences of nature. It would be wrong to intend it as an element of a general epistemological conception. The same laws of *inertia* and of fall of bodies can be perfectly intended as pure consequences of mathematical concepts of speed and force. Such a conception is surely more in line with Lagrange's reductionist attitude, but it makes it very difficult to justify mechanics as a science or *real* natural *phenomena*. Instead of confronting such a difficulty Lagrange prefers—like all scientists of Enlightenment—to conceal it.
119. Cf. *ibid.*, p. 225.
120. Cf. *ibid.*, pp. 226–27. Here motion is clearly represented by space and we have:
    $AB=f(t)$,
    $BC=f(t+\vartheta)-f(t)$ [space traversed along ϑ by a body moving of motion $x=f(t)$],
    $BD=\vartheta f'(t)+(\vartheta^2/2)f''(t)$ [space traversed along ϑ by a body moving of motion $x=f(t)+\vartheta f'(t)+(\vartheta^2/2)f''(t)$],
    $BE=a\vartheta+b\vartheta^2$ [space traversed along ϑ by a body moving of motion $x=f(t)+a\vartheta+b\vartheta^2$].
    Consequently we shall have:

$$\left|f(t+\vartheta)-f(t)-f'(t)\vartheta-\frac{f''(t)}{2!}\vartheta^2\right|=\mathrm{CD}$$

$$|f(t+\vartheta)-f(t)-a\vartheta-b\vartheta^2|=\mathrm{CE}$$

and we can prove that $f(t)+\vartheta f'(t)+(\vartheta^2/2)f''(t)$ is the osculatory parabola to $f(t)$ in t, which represents uniformly accelerated motion with the same instantaneous speed and acceleration of $f(t)$ in t.



121. Cf. Lagrange (1797), pp. 119–22.
122. Cf. *ibid.*, (1797), p. 227.
123. The term "continuous" is used here in the usual modern meaning.
124. Cf. (1797), pp. 227–28.
125. For a justification of my interpretation of fluxional programme cf. Panza (1989), chs. 3 and 4.
126. This is not the case of differential formulation of mechanical *formulæ*. The symbols dy/dx and $y'(x)$ have in fact two different meanings. The first expresses the results of the application of differential algorithm to $y=y(x)$, whereas the second expresses a formal entity (the first Taylor's coefficient) which is absolutely independent from this algorithm.
127. Lagrange uses only one symbols respectively for x, X and X, y, Y and Y, z, Z and Z and uses t to represent both an instant and a time. He is, then, forced to express rectilinear component motions generically by $x=at$, $y=bt$, $z=ct$; $x=(1/2)gt^2$, $y=(1/2)ht^2$, $z=(1/2)kt^2$, introducing the derivative notations x' for a, y' for b, z' for c; x'' for g, y'' for h, z'' for k only at the end.
128. It is clear that each of these motions has to be considered as the result of compositions of three orthogonal motions $x_j=x_j(\tau)$, $y_j=y_j(\tau)$, $z_j=z_j(\tau)$ $(j=1, 2, \ldots, m)$ such that $x_1(t)=x_2(t)=\cdots x_m(t)$; $y_1(t)=y_2(t)=\cdots y_m(t)$; $z_1(t)=z_2(t)=\cdots z_m(t)$.
129. Lagrange limits himself to the case $m=2$ and is not explicit on the analytical deduction of (14) from (12), with the necessary presupposition of the law of composition for codirected speeds.
130. Cf. the note (115).
131. In the present paragraph I shall try to give an interpretation of Lagrange's *formulæ* underlining their relations with variational formulation of the principle of virtual velocities proposed in the *Méchanique analitique*. Even if (20) and (21) are not explicitly deduced by Lagrange in the *Théorie*, they show the complete correspondence between the foundation of mechanics of discrete systems in these treatises.
132. Cf. note (34).
133. Cf. again previous note (115).
134. Cf. Lagrange (1801), pp. 47–52.
135. Cf. Lagrange (1797), p. 60.
136. Cf. *ibid.*, pp. 239–40.
137. I have indicated the derivatives of x relative to v at point w by $x'_v(w)$, $x''_v(w)$, &c..
138. Cf. Newton (1687), pp. 260–69.
139. Cf. J. Bernoulli (1711), pp. 50–1 and N. Bernoulli (1711).
140. Cf. Newton (1713), pp. 232–39. On the whole question cf. section 6, pp. 312–424 of volume VIII

of Whiteside (1967–81).

141. Cf. Lagrange (1797), pp. 241–51. Important changes are in second edition [cf. Lagrange (1813), pp. 334–49]. Clearly it is the "historical" interest of the problem that motivates Lagrange's choice. For a more detailed discussion of Lagrange's solutions and remarks on Newton's first proof, cf. Panza (1988).

142. The matter of this paragraph is treated differently by Lagrange in the first and second edition of *Théorie*. My essential reference is to the second edition. The differences between the two editions are, however, only in the exposition of the matter. Cf. Lagrange (1797), pp. 251–53 and (1813), pp. 350–2.

143. Even if p_j $(j=1, 2, \dots, n)$—being the distances between the origins of forces $u_j(t)$ and point T—are functions of x, y, and z, to intend the (28) as equations of spheric-surfaces we have to suppose these distance as constants. Thus partial derivatives ${}_jS'_x(x, y, z)$, ${}_jS'_y(x, y, z)$ and ${}_jS'_z(x, y, z)$ have to be intended as partial derivatives of ${}_jS(x, y, z)$, respectively relative to variables x, y and z explicitly occurring in these functions; i.e. as partial derivatives of ${}_jS(x, y, z)$ respectively relative to variables x, y and z, being p_j taken as a constant.

144. Cf. Lagrange (1788), pp. 44–9; cf. also (1811–15), pp. 74–9.

145. Cf. Lagrange (1813), p. 240. This result is not made explicit in the first edition, where nevertheless all equations from which it derives are deduced.

146. Cf. Lagrange (1797), p. 254.

147. The term $\Psi_{1,2}(p_{1,2})'$ is obviously null for the considered system. However I have introduced it explicitly in (34) in order to exemplify the general procedure.

148. For the matter of the previous paragraph, cf. Lagrange (1797), pp. 253–56. For the matter of the present paragraph, cf. Lagrange (1813), pp. 352–57.

149. Cf. the previous paragraph I.λ..

150. Cf. Lagrange (1813), p. 356.

151. Cf. Lagrange (1797), p. 256. Obviously we can reach the same equation (40) simply by observing that if the system is completely free to move along the direction of x axis and (30) is the typical form of its equations of condition, then we shall have also

$$\begin{aligned} &f(x_\nu+\xi, y_\nu, z_\nu, x_\mu+\xi, y_\mu, z_\mu, \&c.) \\ &\quad = f(x_\nu, \&c.) + \xi[f'_{x_\nu}(x_\nu, \&c.) + f'_{x_\mu}(x_\nu, \&c.) + \&c.] + \&c. = 0 \end{aligned}$$

from which (40) according to the method of indeterminate coefficients.

152. Cf. the note (151).

Note that in the absence of any external force, we have, for a system free to move along the directions of the three axes,

$$\sum_{i=1}^{n} \mathrm{M}_i[x''_i(t)] = 0; \quad \sum_{i=1}^{n} \mathrm{M}_i[y''_i(t)] = 0; \quad \sum_{i=1}^{n} \mathrm{M}_i[z''_i(t)] = 0$$

from which it is easy to draw:

$$x_G(t) = \mathrm{H}_1 t + \mathrm{H}_2; \quad y_G(t) = \mathrm{K}_1 t + \mathrm{K}_2; \quad x_G(t) = \mathrm{W}_1 t + \mathrm{W}_2$$

(where H_1, H_2, K_1, K_2, W_1 and W_2 are constant), which express the fact that the centre of gravity moves in a uniform rectilinear motion.

154. In these cases the equations expressing the external forces will in fact take the respective forms $\sqrt{z_i - c} - q = 0$ and $\sqrt{x_i^2 + y_i^2 + (z_i - c)^2} - p = 0$ (c, q and p being appropriate constants) and, while in the first case, $x''_i(t)$ and $y''_i(t)$ will be separately null, in the second case, we have:

$$y''_i(t)x_i(t) - z''_i(t)y_i(t) = \sum_{j=1}^{m} \Phi_j \frac{y_i(t)}{p_j} x_i(t) - \Phi_j \frac{x_i(t)}{p_j} y_i(t) = 0$$

155. Cf. the previous paragraph I.τ.

156. It is clear that, being $x_i = x_i(t)$, $y_i = y_i(t)$, $z_i = z_i(t)$ $(i = 1, 2, \dots, n)$, (30) entails $f'_t(x_\nu, y_\nu, z_\nu, \&c.) = 0$.

157. Cf. Lagrange (1797), p. 267.

158. Cf. *ibid.*, p. 276.

159. Cf. Lagrange (1813), p. 381.

160. Cf. Maclaurin (1742).
161. Cf. for example, the third corollary of proposition X of the second book in the first edition of the *Principia* [cf. Newton (1687)] and the second part of the proof itself in the second edition [cf. Newton (1713)].

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